Mirror symmetry

and

Exact Solution of 4D $N = 2$ Gauge Theories $-$ I

Sheldon Katz[a,b], Peter Mayr[c] and Cumrun Vafa[d]

[a] *Oklahoma State University, Stillwater, OK 74078, USA*

[b] *Institut Mittag-Leffler, S-182 62 Djursholm, Sweden*

[c] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*

[d] *Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138,USA*

Using geometric engineering in the context of type II strings, we obtain exact solutions for the moduli space of the Coulomb branch of all $N = 2$ gauge theories in four dimensions involving products of SU gauge groups with arbitrary number of bi-fundamental matter for chosen pairs, as well as an arbitrary number of fundamental matter for each factor. Asymptotic freedom restricts the possibilities to SU groups with bi-fundamental matter chosen according to ADE or affine ADE Dynkin diagrams. Many of the results can be derived in an elementary way using the self-mirror property of $K3$. We find that in certain cases the solution of the Coulomb branch for $N = 2$ gauge theories is given in terms of a three dimensional complex manifold rather than a Riemann surface. We also study new stringy strong coupling fixed points arising from the compactification of higher dimensional theories with tensionless strings and consider applications to three dimensional $N = 4$ theories.

June 1997

1. Introduction

Many non-trivial facts involving exact results for supersymmetric field theories have found their natural explanation in the context of string theories. In particular many exact results about field theories can be obtained from a realization of them by considering special limits of string compactifications and the use of classical string symmetries, such as T-duality or mirror symmetry.

In this paper we concentrate on the case of $N = 2$ quantum field theories in $d = 4$ and obtain exact results for the Coulomb branch of such theories by using classical symmetries and in particular mirror symmetry of type II string compactifications on Calabi-Yau three-folds. Exact $N = 2$ results from string theories were first obtained in [1] by conjecturing an exact duality between heterotic strings on $K3 \times T^2$ and certain Calabi-Yau compactifications of type II strings. It was pointed out in [2] that the relevant Calabi-Yau's which arise in these dualities are of the form of a $K3$ fibration over a 2-sphere. It was argued in [3] that this has a natural interpretation based on fibering the duality of heterotic string on $K3$ with type IIA strings on T^4. In fact it was shown in [4] that $K3$ fibration of Calabi-Yau threefold is a necessary condition for any $N = 2$ type II/heterotic duality. After a careful study of the field theory content of the string duality conjectured in [1] it was found in [5] that the relevant field theory part of the moduli comes from an ADE type singularity of the $K3$ fibered over the sphere $\mathbf{P^1}$. This resulted in a derivation of many of the exact results known for $N = 2$ field theories in four dimensions [6][7]. Moreover it was shown in [8] (for a review of various aspects of this construction see [9]) how to use this string description and compute directly the BPS spectrum of $N = 2$ strings in terms of geodesics on the Seiberg-Witten curve. This study of BPS states has now been extended to many cases [10]. It was also shown in [8] using the T-duality between NS 5-branes and ADE singularities [11] that one can view the four dimensional theory as coming from the study of the type IIA 5-brane whose worldvolume is $R^4 \times \Sigma$ where Σ is a non-compact version of the Seiberg-Witten curve.

The analysis in [8] hinted that the basic idea of embedding $N = 2$ field theories in type II compactifications on Calabi-Yau threefolds, can be done *without appealing to any non-trivial string duality.* Namely just using the fact that in type IIA D2 branes wrapping over vanishing spheres of ADE singularity give rise to gauge symmetries in six dimensions, and fibering that over a sphere giving $N = 2$ theories in four dimensions gives by itself a complete picture. This idea was further developed in [12] where it was called *geometric*

engineering of quantum field theories. In that context one starts with type IIA strings on Calabi-Yau threefolds, performs a local mirror transformation, and ends up with a local type IIB model. Moreover the moduli of the complex structure mirror type IIB is not corrected quantum mechanically and thus gives the exact quantum answer already at the type IIB string *and* the worldsheet tree level. The main aim of this paper is to continue the idea of geometric engineering into a much wider class of $N = 2$ theories. Since there are many possibilities to consider, we have decided to devote two papers to this subject. In the first paper (the present one) we discuss geometric engineering of $N = 2$ gauge theories involving products of SU groups with bi-fundamental matter, as well as extra fundamental matter. We provide exact solution for the Coulomb branch moduli for *all* such cases. The various cases will depend on the configuration of matter we consider which allows us to attach a "quiver diagram" associated to our theory. In particular we consider a diagram where for each SU gauge group we consider a node, and for each pair of groups with bi-fundamental matter, we connect the corresponding nodes with a line. It turns out, as we will show, that asymptotic freedom in four dimensions implies that the corresponding diagram corresponds to ADE Dynkin diagram or affine ADE Dynkin diagram. In the case we get ADE Dynkin diagram we can add extra fundamental matter to make the theory superconformal. In the case of affine ADE the condition of having asymptotic freedom will imply that the rank of each SU gauge group is correlated with the Dynkin number on the corresponding node, and that automatically leads to a superconformal theory (without any extra fundamental matter). As we will show, the S-duality group in all these cases corresponds to the fundamental group of the moduli of flat ADE gauge fields on elliptic curve or the degenerate elliptic curve, depending on whether we are dealing with the affine ADE Dynkin diagram or the ordinary ADE Dynkin diagram. It is quite surprising from the field theory perspective how the ADE gauge fields appear in this story. We will explain its stringy origin.

We construct the type IIA geometry and its type IIB mirror for all such theories. The complex geometry of the mirror, which gives the exact solution for the moduli of Coulomb branch, is generally given by a non-compact piece of a Calabi-Yau 3-fold. Sometimes, *but not always*, we find that the data can be reduced to a Riemann surface Σ, which as noted in [8] becomes equivalent by a T-duality [11] to the fivebrane of type IIA (or M-theory) on $R^4 \times \Sigma$.

In a subsequent paper we generalize these constructions to include more general gauge groups and matter content.

The organization of this paper is as follows: In section 2 we introduce the basic idea of geometric engineering of $N = 2$ theories in type IIA strings and the strategy of getting exact results by application of mirror symmetry. We also show how asymptotic freedom restricts the bi-fundamental matter structure to be in the form of ADE or affine ADE Dynkin diagrams. In section 3 we discuss some relevant intuitive aspects of mirror symmetry, and in particular the fact that ALE space is self-mirror, to help us develop an intuition about the results that are to follow. Moreover, in that section we give a simple, but heuristic derivation of some of the results (which are derived later in the paper using more sophisticated and rigorous toric methods). These include the case of SU gauge groups arranged along a linear chain with bi-fundamental matter between nearest neighbors (the A case), and the case of SU with bi-fundamental matter according to links of affine E Dynkin diagram. In section 4 we discuss aspects of toric geometry and its relation to mirror symmetry. The discussion in this section is aimed at putting the intuitive arguments in section 3 in the powerful setup of toric geometry. We aim to provide an essentially self-contained introduction to toric geometry and its relation to mirror symmetry, emphasizing aspects which we will use in this paper. In section 5 we revisit the case of SU gauge groups arranged along the linear chain (the A case) with bi-fundamental matter and rederive the results of section 3 more rigorously. This also allows us to give more information about the solution including the relevant meromorphic 1-form on the Riemann surface. In section 6 we show how arbitrary matter in the fundamental representation can be added for each gauge factor. This construction will be applicable to all ADE cases. In section 6 we mainly concentrate on the A case with extra matter. In section 7 we discuss a quiver configuration corresponding to a trivalent vertex. This in particular allows us to rederive the results concerning the affine E cases discussed in section 3 more rigorously. We also show how the theories based on ordinary Dynkin diagrams of D and E are realized in this construction. In section 8 we give a uniform treatment for quivers based on affine ADE Dynkin diagrams, by studying the mirror of elliptic ADE singularities in 2-complex dimensions (this gives the first derivation for the affine A and D cases and a third derivation for the affine E case as the quiver). We also discuss the S-duality group for these affine ADE cases as well as the case based on ordinary ADE Dynkin diagrams. In particular we find that the S-duality groups for all these cases have a rather simple description in terms of the duality group for ADE flat connections on a two dimensional torus and its degeneration. In section 9 we discuss some realization of certain stringy strong coupling fixed points, which can be solved using our method. In particular we show that in some

cases we get new critical $N = 2$ theories (which are related to toroidal compactifications of some tensionless non-critical string theories found in six dimensions). In section 10 we apply our results to obtain some new results for $N = 4$ theories in $d = 3$. In particular we construct the dual of k small instantons of E_8 theory compactified to three dimensions, extending the result for $k = 1$ obtained in [13] and verifying the conjecture in [14].

2. Basic Setup

Our starting point is a local model for type IIA compactification on a non-compact Calabi-Yau threefold. We first need to review some facts about properties of type IIA strings on ALE spaces (some of which arise as local singularities of $K3$ manifold). Consider an ALE space with an ADE type singularity. For simplicity let us consider the A_1 case. In this case we have a singularity of the form

$$xy + z^2 = 0. \tag{2.1}$$

This singularity can be resolved by 'blowing up'. Concretely what this means is that we consider a new variable

$$\tilde{x} = x/z$$

which implies that if we substitute it into the above equation it is of the form

$$\tilde{x}y + z = 0$$

which is not singular any more. This resolution has been at the price of doing a singular change of variables. Mathematically what we have done is to replace $x = z = 0$ which was the point singularity of the original space by a whole sphere parametrized by $\tilde{x}$, and having done that we have avoided the singularity. We are only describing the complex structure of the curve, but if we wish to put metrics to make this resolution continuously match with the singular manifold we started with, we have to make the sphere denoted by $\tilde{x}$ have zero volume at the beginning and then increase it continuously to a finite value. In the context of type IIA string propagating on this background, D2 branes wrapped around the $\tilde{x}$ sphere will give a vector particle whose mass is proportional to the volume of the blown up $\mathbf{P}^1$ (2-sphere). Actually we can have two different orientations for the wrapping of the D2 brane and so we obtain two states, which we will denote by $W^\pm$. The states $W^\pm$ are charged under the $U(1)$ gauge field corresponding to decomposition of the type

4

IIA 3-form in terms of the harmonic form on the $\mathbf{P}^1$. Let us call this vector field by Z. In the limit where the $\mathbf{P}^1$ shrinks we get three massless vector fields $W^{\pm}, Z$ which form an $SU(2)$ adjoint. The story is similar for the general ADE singularities where we obtain an enhanced ADE gauge symmetry in the limit where all the 2-cycles shrink (for a recent review of ADE blowups in the physics literature see [13]). We thus obtain an $N = 2$ ADE gauge symmetry in $d = 6$.

If we compactify on a T^2 down to $d = 4$ we obtain an $N = 4$ system. Note that the extra scalars we get in the $N = 4$ system can be identified with the expectation values of Wilson lines on the T^2. We are however interested in obtaining an $N = 2$ system in $d = 4$. In order to kill the extra scalars we need the intermediate two space to have no cycles, which means that we need a 2-sphere. Mathematically what this means is that we have a three complex dimensional fibered space with a two sphere as the base and the ALE space as the fiber. The structure of the fibration is such that the whole three dimensional non-compact space can be viewed as a non-compact Calabi-Yau threefold. We have thus *engineered* $N = 2$ pure Yang-Mills theory in $d = 4$. Note that the volume of the base $\mathbf{P}^1$ is related to the coupling constant of Yang-Mills in $d = 4$ by the usual volume factor, namely

$$ V_{base} = \frac{1}{g^2} \tag{2.2} $$

The Coulomb parameters of the $N = 2$ system in $d = 4$ get mapped to the sizes of the blown up $\mathbf{P}^1$'s in the fiber. Sometimes when we refer to the fiber geometry we only concentrate on the compact parts of it, namely the blown up spheres.

2.1. Incorporation of Matter

There have been a number of works which relate how matter arises from the geometry of Calabi-Yau compactifications. We will follow the approach in [15] which itself was based on earlier works [16][17]. For concreteness let us explain how we can obtain bi-fundamental hypermultiplets of $SU(n) \times SU(m)$. Suppose we have an A_{n-1} singularity over $\mathbf{P}^1$ and an A_{m-1} singularity over another $\mathbf{P}^1$. Moreover the two $\mathbf{P}^1$'s meet at a point where the singularity jumps to an A_{n+m-1} singularity [18]. Let $z_{1,2}$ denote the coordinates of the two $\mathbf{P}^1$'s and assume that the intersection point is at $z_1 = z_2 = 0$. Consider the threefold which is locally given by

$$ xy = z_1^m z_2^n $$

5

Then for arbitrary $z_1 \neq 0$ we have an A_{n-1} singularity and for arbitrary $z_2 \neq 0$ we have an A_{m-1} singularity. Let us change variables to $z_2 = z_1 + t$. The above equation then becomes

$$xy = z_1^m (z_1 + t)^n = z_1^{n+m} + \ldots + t^n z_1^m$$

This can be reinterpreted as an $SU(n+m)$ singularity in 6 dimensions which is broken to $SU(n) \times SU(m) \times U(1)$ by giving space-dependent vevs to some of the scalars in the Cartan of $SU(n+m)$. Recall that if we had just compactified $SU(n+m)$ on T^2 we would have gotten an $N = 4$ system, which in the $N = 2$ terms contains an extra hypermultiplet in the adjoint. Out of the adjoint hypermultiplet of $SU(n+m)$ the above space dependent breaking of $SU(n+m) \to SU(n) \times SU(m) \times U(1)$ gives rise to $(n, \bar{m})$ of $SU(n) \times SU(m)$ charged under the $U(1)$ and localized near $z_1 = z_2 = 0$, as explained in [15] (see also the closely related case [19]). The adjoint of $SU(n) \times SU(m)$ that one would get in this picture has a mass because of the global geometry of the base $\mathbf{P}^1$'s as explained before. So in this way we have engineered $SU(n) \times SU(m) \times U(1)$ with bi-fundamental matter charged also under $U(1)$. It may appear that we are getting an 'unwanted' $U(1)$. This is not quite true. In fact we need to be able to give an arbitrary mass to the bi-fundamental matter, and this can be done by going to the Coulomb branch of $U(1)$. Since the $U(1)$ is not asymptotically free we can ignore its infrared dynamics and just think about its Coulomb branch as the mass parameters of the $SU(n) \times SU(m)$ system.
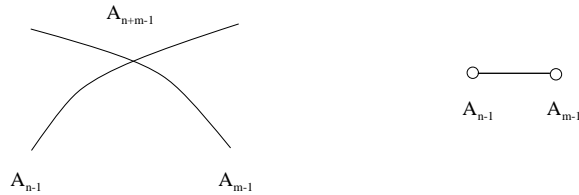

Fig. 1: At the intersection of two base curves carrying A_{n-1} and A_{m-1} fibers, an A_{n+m-1} singularity develops, with the extra 2-sphere supporting the bi-fundamental matter. This is denoted by a link in the quiver diagram on the right.

It is now straightforward to generalize this to arbitrary product of SU groups with matter in bi-fundamentals. The data for such a theory can be drawn in terms of a graph, where to each gauge group we associate a node (vertex) in the graph and for each bi-fundamental matter between pairs of groups we draw a line connecting the corresponding nodes. Geometrically we engineer this theory by associating to each node a base $\mathbf{P}^1$ over which there

is the corresponding SU singularity, and to each pair of nodes connected, we associate an intersection of the base $\mathbf{P}^1$'s, where over the intersection point the singularity is enhanced to an $SU(n + m)$ (assuming the nodes correspond to $SU(n)$ and $SU(m)$ groups). Note that if we are interested in addition in getting fundamental matter for each group, this can be done by adding extra SU groups with bi-fundamental matter, roughly by gauging the flavor group, and making the coupling constant of the extra flavor group weak, by making the base of the corresponding $\mathbf{P}^1$ big (recall (2.2)). This process we sometimes call as adding extra nodes and 'degenerating' them.

Clearly we can generalize this to more general groups in the fiber (such as D, E and non-simply laced groups [20] [21]) and more general kind of matter (coming from the breaking of an adjoint of a higher group) as discussed in [15]. This more general situation will be the subject of an upcoming paper [22].

2.2. Restrictions from Asymptotic Freedom and ADE Dynkin Diagrams

As noted above we are considering the case of product of $SU(k_i)$ gauge groups with bi-fundamental matter between some pairs and some extra fundamentals for each group. For interesting four dimensional field theories, one would be interested in theories with negative β-function for all gauge factors[1]. This turns out to put a severe restriction on the choice of the bi-fundamental matter one chooses. As discussed before the structure of bi-fundamental matter gives rise to a graph where for each node i of the graph we consider an $SU(k_i)$ gauge group of some rank k_i , and for each pair of bi-fundamentals between the i-th and j-th group we draw a line between the i-th and j-th node. We will now show that the corresponding graph is that of ADE Dynkin diagram or its affine extension. In other words, quite independently of what extra fundamental matter one has, the geometry of the bi-fundamental matter is already very restrictive[2].

Suppose we have r gauge group factors. Let M be a symmetric $r \times r$ matrix with diagonal entries $M_{ii} = 2$ and off-diagonal entries $M_{ij} = -N_{ij}$ where N_{ij} is the number of bi-fundamentals between i-th and j-th gauge groups. Let k denote the vector which gives

---

[1] We will nevertheless also consider cases with positive beta function in four dimensions, since these four-dimensional field theories give rise to interesting three dimensional field theories which are asymptotically free after further compactification on a circle [13].

[2] We thank Noam Elkies for providing the mathematical argument that follows. See also [23].

the rank of the gauge groups. Then the condition of asymptotic freedom can be succinctly stated as the requirement

$$(Mk)_i = \sum_j M_{ij} k_j = 2k_i - \sum_{j \neq i} N_{ij} k_j \geq 0.$$

In other words Mk is a positive semi-definite vector. Now we need the following fact known as Perron-Frobenius theorem:

If S is a symmetric matrix with positive entries, then the eigenvector v corresponding to its maximal positive eigenvalue can be chosen to have positive entries.

To prove this, let us normalize v such that $v^t v = 1$. Then v satisfies the condition that $v^t S v$ is maximal subject to $v^t v = 1$. Consider a positive vector $v' = |v|$. Note that $v'^t v' = 1$. Since S has positive entries, we deduce

$$v'^t S v' \geq v^t S v.$$

But since v maximizes $v^t S v$ then the above must be an equality and so v' should be the same as v (up to an overall sign). In other words the eigenvector corresponding to the maximal eigenvalue can be chosen to correspond to a positive vector.

Now let us apply this theorem to the matrix N, which is a positive symmetric matrix. Since $M = 2I - N$ where I is the identity matrix, we learn that if v is a positive vector corresponding to the maximal eigenvalue of N, then it is also the smallest eigenvalue of M. Let us call this eigenvalue by λ, i.e. $Mv = \lambda v$. Let us consider

$$v^t M k$$

Since $Mk \geq 0$, by assumption of $\beta \leq 0$, and since v is a positive vector we learn that $v^t M k$ is positive. Thus we have

$$0 \leq v^t M k = \lambda(v^t k).$$

Since both v and k are positive vectors this implies that $\lambda \geq 0$. Since λ, the smallest eigenvalue of M, is positive this implies, as is well known, that M corresponds to an ADE Dynkin diagram if $\lambda > 0$ or corresponds to an affine ADE Dynkin diagram if $\lambda = 0$. Moreover, if M corresponds to affine ADE Dynkin diagram, since $v^t M k = 0$ and $v^t > 0$ this implies that $Mk = 0$, i.e. we learn that k is proportional to v which is the only eigenvector corresponding to zero eigenvalue for affine ADE generalized Cartan matrix. Note that this implies that k is a vector which is proportional to the vector of Dynkin numbers associated

8

to the nodes of affine Dynkin diagram. These interesting special cases correspond to having ranks k_i of the i-th A factor be given by a common integer multiple of the Dynkin labels s_i of the affine ADE $k_i = s_i n$. In case we have ordinary ADE Dynkin diagram there is no choice of rank vector k which makes the theory superconformal just by having bi-fundamental matter, because $Mk > 0$. In such cases we can add Mk extra fundamental matter to the corresponding gauge group and make the theory superconformal. These cases we will consider in more detail later. The group $\hat{G}$ associated to the base geometry will turn out to be of physical relevance in many respects, such as determining the S-duality group of the conformal four-dimensional theory, as we will discuss later in the paper.

2.3. Strategy in Extracting Exact Results: Mirror Symmetry

We have described how we can geometrically engineer $N = 2$ quantum field theories in four dimensions, in particular for SU groups with bi-fundamental matter, in the context of type IIA strings. We are interested in using this geometry to learn about gauge dynamics. In general we have Higgs and Coulomb branches for $N = 2$ theories. The Higgs moduli correspond to the moduli of scalars in the hypermultiplets whereas the Coulomb branches correspond to the moduli of scalars in the vector multiplets. Moreover there are no (F-type) mixtures between hypermultiplets and vector multiplets and so the two do not mix with each other. Whereas the Higgs branches are easily computable using classical Lagrangians of gauge theory, the same is not true for the Coulomb branch which receives non-perturbative point-like instanton corrections. We are interested in computing these corrections. The simplification occurs in type II theories on Calabi-Yau because the string coupling constant is in a hypermultiplet (see [24] for a careful treatment). Since the geometry of the Coulomb branch is independent of hypermultiplet vevs, we can take an arbitrary string coupling without changing the answer [25]. This implies that if we compute the tree level answer for Coulomb branch in string theory it is the exact answer (this was in fact used in [1][26]). We thus need to know the classical answer for the Coulomb branch in type IIA string propagating on the local geometry we have constructed. However the classical answer on the type IIA side *does* receive worldsheet instanton corrections. In fact this construction maps the contribution of spacetime instantons of gauge theory to special growth of the number of instantons of a two dimensional worldsheet theory, as discussed in [12]. Mirror symmetry comes to the rescue and results in summing up the worldsheet instantons by giving a local mirror Calabi-Yau geometry which gives an identical theory where we now consider type IIB strings instead of type IIA. In this case there are *no*

worldsheet corrections and thus the exact gauge theory answer can be read off from a classical computation of a 2-dimensional theory. This is thus our strategy: Find the mirror of the geometry we have engineered and then extract the exact quantum answers of the gauge system by classical computations. We thus see that mirror symmetry is a key fact allowing us to extract exact answers. We now turn to a review of certain aspects of mirror symmetry.

3. Intuitive Aspects of Mirror Symmetry and a Simple Derivation of Exact Results

In this section we will review some aspects of mirror symmetry which provides an intuitive basis for the results which will follow later in the paper. Moreover in this section we give a simple but less rigorous derivation of some of our basic results that will be rederived more rigorously using toric methods in later sections. The cases we will derive the exact results for in this section include the case of linear chain of SU groups and SU groups corresponding to the affine $E_{6,7,8}$ quiver diagrams.

Consider a d-dimensional complex Calabi-Yau manifold M and its mirror pair W. This in particular means that for d odd type IIA(B) on M is equivalent to type IIB(A) on W where the role of Kähler deformations and complex deformations get exchanged. If d is even, type IIA(B) on M is equivalent to type IIA(B) on W, again with the role of Kähler and complex deformations exchanged. Complex dimensions 1 and 2 are very special because there are very few Calabi-Yau manifolds. In dimension one there is only T^2 and in dimension 2 there is only $K3$ (apart from T^4 which has trivial holonomy). This scarcity in low dimensions in particular leads to the fact that T^2 and $K3$ are self-mirror. The case of T^2 is very well known and is a simple consequence of T-duality. In the case of $K3$ this is also a true but less trivial fact [27]. Even though we will eventually be interested in the case of complex dimension 3, aspects of $K3$ and its self-mirror property play a crucial role in this section and so we will now discuss it in a bit more detail.

As already discussed, we will only be interested in a local model of $K3$ with singularities. Let us recall that the singularities one encounters in $K3$ are of ADE type. Our local model will consist of ALE space of ADE type and we are interested in constructing the mirror. Let us consider an A_{n-1} ALE space. This can be described by a singular complex 2-manifold whose complex structure is given by

$$xy + z^n = 0, \qquad (3.1)$$

where x, y and z are complex numbers. This space is singular at the origin. There are two ways this singularity can be remedied: We can either deform the defining equation to make it less singular or we can 'blow up' the singularity. The deformation which involves changing the complex structure is given by

$$xy + \prod_{i=1}^{n}(z - a_i) = 0,$$

with distinct a_i. Up to a shift in z there are $n - 1$ physical parameters defining this deformation. On the other hand we can resolve the singularity by keeping the same defining complex equation (3.1) but by 'blowing up' the singularity, introducing $n - 1$ extra spheres which intersect one another in the way dictated by the Dynkin diagram of A_{n-1}. This blow up is specified by $n-1$ complex parameters, corresponding to the size and the B-field on each sphere. In the blow up one is varying the Kähler classes on the ALE space. Mirror symmetry in this case states that if we are interested in studying type IIA(B) on this blow up space it is equivalent to studying type IIA(B) on the complex deformed space exchanging the $n - 1$ complex parameters corresponding to the complexified Kähler classes with $n - 1$ complex parameters describing deformation of the defining equation.

In the applications we will consider it is also important to consider the case where the local model for $K3$ involves an elliptic fibration. This is a well known subject mathematically [28] [18]. In particular the elliptic fibration over the plane can develop ADE singularities as we change the complex structure of the $K3$. Again we can blow up the singularity and we obtain new 2-cycles, a basis of which can be taken to be $\mathbf{P}^1$'s which intersect according to the Dynkin diagram of the ADE group, as was the case above. The only new ingredient in the elliptic case is that there is an extra special 2-cycle class, whose intersection with the other cycles can be represented by an extra node making the Dynkin diagram an *affine* Dynkin diagram. If s_i denote the Dynkin numbers associated with each node of the Dynkin diagram, and if we denote the i-th 2-cycle class by C_i the 2-cycle class of the elliptic fiber E can be represented by

$$E = \sum s_i C_i \tag{3.2}$$

Note that this is consistent with the fact that $E \cdot E = 0$. The extra cycle corresponding to the extra node on the affine Dynkin diagram is of finite size even after all the other cycles have shrunk. This follows from the fact that when all the other cycles shrink the relation

11

(3.2) implies that the size of the extra 2-cycle corresponding to the affine node is the same as the size of the elliptic fiber (recall that the Dynkin number for the affine node is 1).

Mirror symmetry implies that the Kahler deformation of the blow up is equivalent to complex deformation of the mirror geometry. The complex deformations of the mirror has in turn another description which will prove useful for us. Consider type IIA string on a 2 dimensional complex space with elliptic ADE singularity. If we compactify further on another T^2 we obtain an $N = 4$ theory in $d = 4$. From the viewpoint of $N = 1$ theory we have three adjoint chiral fields X, Y, Z and a superpotential

$$W = \text{Tr}[[X, Y], Z]$$

The Higgs branch of $N = 4$ theory can be viewed as giving vevs to the Cartan of X or Y or Z. In fact a $U(3)$ subgroup of the $SO(6)$ R-symmetry group rotates these fields among each other. We can identify X with the blowing up of the elliptic ADE singularity, Y with the deformation of the singularity and Z as giving Wilson lines to the ADE gauge group on the compactified T^2. Given the R-symmetry we deduce that three deformations are equivalent and give rise to the same moduli space. Thus we conclude, in particular, that *the moduli space of blowups of elliptic ADE singularities of complex surface is (mirror to) the moduli space of flat ADE connections on a T^2*. This result will be important later when we discuss S-dualities that arise in field theories we study. We will give an alternative derivation of this fact in section 8.

3.1. Base Geometry vs. Fiber Geometry

So far we have discussed mirror symmetry in complex dimension 2. However for the purposes of the present paper we are actually interested in the case of complex dimension 3. The two are not unrelated, when we recall that we are interested in fibering an A-type singularity over some collection of $\mathbf{P}^1$'s. So roughly speaking all we have to do is to *also apply mirror symmetry to the base as well*. However, as we have discussed before, the interesting class of configurations of the base also correspond to when we have base $\mathbf{P}^1$'s which intersect according to the ADE or affine ADE Dynkin diagrams. But we have already discussed how these also arise in the complex 2-dimensional case. So *the mirror to both the base and the fiber geometry will involve aspects of two dimensional mirror symmetry* already discussed. The only non-trivial data is how a particular configuration of fiber $\mathbf{P}^1$'s over the base $\mathbf{P}^1$'s is translated to mixing these two mirror symmetry transformations. We will now try to develop this intuitively to arrive at a heuristic derivation of some of the results which we will derive more rigorously later.

3.2. An Intuitive Derivation of Linear Chain of SU Groups

Let us consider the case of engineering of a linear chain of SU gauge groups arranged along a line, with bi-fundamental matter between the adjacent groups. Let us suppose we have m gauge groups. This means in particular that the base geometry is A_m. As discussed before the mirror of this base geometry is given by

$$P_{m+1}(z) + uv = 0 \tag{3.3}$$

where $P_{m+1}(z)$ denotes a polynomial of degree $m+1$ in z which is mirror to blowing up the A_m singularity. The fiber geometry will be a combination of A_{n-1}'s where n varies over each $\mathbf{P}^1$ in the base depending on the arrangement of the SU groups along the linear chain. If we denote the fiber variable w (which together with u, v, z and an equation give a threefold), for each of the $SU(n)$ factors in the fiber we expect to have a polynomial $P_n(w)$ of degree n in w. This should clearly be correlated with the coefficients in (3.3), because the blowing up of the base geometry is mirrored to complex deformation of the equation. Suppose the first group along the chain is $SU(n_1)$. Then we should see this group when we blow up the base only once, which is mirror to

$$P_{m+1}(z) = z^{m+1} + az^m,$$

where the A_m singularity has been reduced to $A_m \to A_{m-1}$. At this point we should be able to see the fiber mirror because we have blown up the base once and $SU(n_1)$ is the singularity supported on the first blowup. This implies, applying the mirror symmetry now to the fiber, that the coefficient a in the above equation should be a polynomial of degree n_1 in w, i.e. we have for the defining equation of the threefold

$$z^{m+1} + P_{n_1}(w)z^m + uv = 0.$$

Now we introduce the next blow up in the base which changes the above equation to

$$z^{m+1} + P_{n_1}(w)z^m + bz^{m-1} + uv = 0.$$

The last $\mathbf{P}^1$ that we have blown up is reflected in the coefficient of the smallest power of z being non-zero. Thus just as before, we now expect b to be a function $P_{n_2}(w)$ of degree n_2 in w. Continuing this reasoning we will end up with the local model for the threefold

$$\sum_{k=1}^{m+1} z^k P_{n_{m+1-k}}(w) + uv = 0,$$

13

where we have put $P_{n_0}(w) \equiv 1$. Note that the case of one gauge group (i.e. $m = 1$) was already considered in [8][12] which agrees with the above result. Moreover it was noted in [8] that one can use the T-duality which relates C^* fibrations in type IIB with NS 5-branes of type IIA (and vice versa) [11] to show that this is equivalent to considering an NS 5-brane of type IIA whose worldvolume is $\Sigma \times R^4$ and where Σ in this case is a Riemann surface with equation

$$\Sigma: \qquad \sum_{k=1}^{m+1} z^k P_{n_{m+1-k}}(w) = 0$$

carved out of the (w, z) space. Moreover it was noted in [8] that the relevant metric on Σ is provided by the SW meromorphic 1-form on Σ. This in particular shows that Σ is non-compact. Recently this result of [8] for one gauge group was rederived from the imbedding of type IIA branes in M-theory in [29] and extended to the case of linear chain (with arbitrary number of fundamentals)[3]. That result agrees with what we have found above. Later on in this paper we will generalize our derivation to the linear chain of A-groups and in addition with arbitrary number of fundamentals for each group and we fully recover the results of [29] from perturbative symmetries of strings.

3.3. An Intuitive Derivation for affine E as the Base

So far we only considered linear chains. As discussed before an interesting case involves the configuration of the A-groups arranged along the affine ADE Dynkin diagrams, where the rank of the SU gauge group is proportional (with fixed proportionality) to the Dynkin number of the corresponding node. In this case we get $N = 2$ superconformal theories. Here we show how the general result for the Coulomb branch of A-groups arranged according to the affine E as the base geometry can be obtained.

Just as in the linear chain considered above we first need to know the mirror for the base geometry. In particular we need a complex geometry whose deformation is mirror to blowing up elliptic E-singularities. To do this we recall that there are three special constructions of $K3$ given by orbifolds which give rise to elliptic E_6, E_7 and E_8 singularities:

$$K3 = \frac{T^2 \times T^2}{Z_3} \qquad E_6 \quad \text{singularity,}$$

$$K3 = \frac{T^2 \times T^2}{Z_4} \qquad E_7 \quad \text{singularity,}$$

---

[3] For an extension of these methods to other gauge groups, see [30].

$$K3 = \frac{T^2 \times T^2}{Z_6} \qquad E_8 \quad \text{singularity,}$$

where the T^2's are special, in that in the first and third case above they correspond to hexagonal lattice, and in the second case they correspond to square lattice (these singularities were studied in the context of F-theory compactifications in [31] generalizing the work [32]). We are interested in finding the mirror for these constructions. Actually we will only be interested in the limit where one of the T^2's is replaced by $\mathbf{C}$, the complex plane, where an isolated elliptic singularity appears.

Let us first consider the E_6 case. In this case the mirror is given by the LG theory (modded out by an overall Z_3) with the superpotential (see [33])

$$W = x^3 + y^3 + z^3 + axyz + x'^3 + y'^3 + z'^3 + a'x'y'z' + \text{deformations.}$$

Here the unprimed variables denote the mirror to one torus and the primed variables the mirror to the other. Also the deformation monomials are of total degree 3 and mix up the unprimed and primed variables. There are three fixed points on each torus which get identified with monomials x, y, z and x', y', z' and blowing up various fixed points, is mirror to choosing the combination of corresponding monomials as deformation of the above potential. Recalling that we are interested in the limit where the primed torus becomes infinitely big (corresponding to sending $a' \to \infty$) and concentrating on one fixed point on the $\mathbf{C}$ plane, corresponding say to the variable x', the deformation monomials will be of the form

$$x'^3, x'^2(x, y, z), x'(x^2, y^2, z^2).$$

Thus going to the patch where $x' = 1$ and ignoring y' and z' which play no role in the above deformations we find that the relevant deformation is given by the geometry

$$x^3 + y^3 + z^3 + axyz + (bx^2 + cy^2 + dz^2) + (ex + fy + gz) + h = 0.$$

Thus we have found the mirror to blowing up elliptic E_6 singularity, where the monomials $1, x, x^2$ correspond to the blown up $\mathbf{P}^1$'s of one fixed point on the T^2 and correspond to one edge of affine E_6 Dynkin diagram starting from the trivalent vertex of affine E_6, and similarly for $1, y, y^2$ and $1, z, z^2$. Now we consider the fiber in addition to this which will correlate with these monomials just as in the case of the linear chain with the fiber

geometry. Basically we apply the idea of incorporating the fiber in the linear chain case to each straight edge of affine Dynkin diagram. We find the local threefold is now given by

$$0 = x^3 + y^3 + z^3 + axyz + P_{3k}(w) +$$

$$\sum_{i=1}^{2} P_{i \cdot k}^x(w) x^{3-i} + \sum_{i=1}^{2} P_{i \cdot k}^y(w) y^{3-i} + \sum_{i=1}^{2} P_{i \cdot k}^z(w) z^{3-i}$$

(it is also easy to write the geometry for the more general case where the β-function is not zero, by choosing polynomials of different degrees than those considered above, just as in the linear chain case). The generalization to the case where the base geometry is affine E_7 or E_8 is straight-forward, where we start with the elliptic curve $y^2 + x^4 + z^4$ for the E_7 case and $y^2 + x^3 + z^6$ for the E_8 case. The connection with blowing up of the $T^2 \times C/Z_4$ and $T^2 \times C/Z_6$ are similar to the previous case. In particular for E_7 the monomials x, z correspond to blowing up the fixed point of order 4 whereas y corresponds to blowing up the fixed point of order 2. In the case of E_8, z corresponds to the fixed point of order 6, x corresponds to the fixed point of order 3 and y corresponds to the fixed point of order 2. It is also easy to include the fiber geometry as in the case of linear chain. For the E_7 case with vanishing β-function we have the mirror geometry

$$x^4 + z^4 + y^2 + axyz + \sum_{i=1}^{3} P_{i \cdot k}^x(w) x^{4-i} + \sum_{i=1}^{3} P_{i \cdot k}^y(w) z^{4-i} + P_{2k}^z(w) y + P_{4k}(w) = 0$$

and for the E_8 case we obtain

$$x^3 + y^2 + z^6 + axyz + \sum_{i=1}^{2} P_{2i \cdot k}^x(w) x^{3-i} + P_{3k}^y(w) y + \sum_{i=1}^{5} P_{i \cdot k}^z(w) z^{6-i} + P_{6k}(w) = 0.$$

Note that for these cases *we cannot reduce the data solving the Coulomb branch of the $N = 2$ system to a Riemann surface.* However we still have an equally useful description of the Coulomb branch in terms of three dimensional Calabi-Yau manifolds given above. In particular one has a holomorphic 3-form whose periods give the central terms in the $N = 2$ SUSY algebra, and one can read off the effective coupling constants of the gauge theory from these periods, just as would be the case for the $N = 2$ solutions associated to Riemann surfaces.

4. Toric Geometry and Linear Sigma Models

In the previous section we have seen how heuristic applications of mirror symmetry goes a long way in giving the type IIB geometry dual to a given type IIA geometry of Calabi-Yau threefolds. However for more general cases and also to prove more rigorously the assertions of the previous section, we need to recall in more detail some of the machinery needed for this purpose [34][35]. We first have to construct a 2 dimensional quantum field theory which describes the propagation of type IIA strings in the local model which we are interested in. Next we have to use this to construct the mirror geometry. What we will do now is to show how type IIA geometry can be summarized in terms of *toric data* and how this can be used to construct the relevant mirror.

In physical terms the easiest way to construct the type IIA background is in terms of linear sigma models [34]. This involves considering an $N = 2$ gauge system with some matter which in the infrared describes the conformal field theory corresponding to the string propagation in a desired background. For our purposes it suffices to consider the case with gauge group $U(1)^r$, with k matter fields x_i. One can also consider adding superpotential terms involving the fields, and we shall need that for later applications. There are also r FI D-terms we can add to the theory, one for each $U(1)$. Let q_i^a denote the a-th $U(1)$ charge of the x_i field. The condition that the theory has an extra R-symmetry (which can thus flow to a non-trivial conformal theory) is that

$$\sum_i q_i^a = 0. \tag{4.1}$$

To start with, which is sufficient for some of the applications, we will consider a theory with no superpotential. The vacuum configurations for this theory is described by the gauge invariant fields. It is well known that this is the same as the manifold

$$\mathbf{C}^k/(\mathbf{C}^\star)^r, \tag{4.2}$$

where $\mathbf{C}^k$ corresponds to the complex values for x_i and where the a-the $\mathbf{C}^\star$ action is given by

$$x_i \to x_i \lambda^{q_i^a}. \tag{4.3}$$

The manifold (4.2) is the geometry which the linear sigma model produces. Note that the complex dimension of this manifold is $d = k - r$. This geometry is generically singular and putting the FI D-terms into the Lagrangian has the effect of resolving the singularity by blowing up the manifold. Just as in the previous section, let us first concentrate on the case of complex dimension 2.

4.1. Type IIA on A_{n-1} Background and Toric Geometry

Let us consider our first concrete example. We will find the linear sigma model for A_1 singularity of $K3$. For this purpose it is sufficient to consider the case of $U(1)$ gauge theory with three matter fields x_i , whose charges are given by

$$l^{(a)} \equiv (q_1^a, \dots, q_k^a) = (1, -2, 1). \tag{4.4}$$

The gauge invariant geometry (using (4.2)) associated to this is obtained by considering the generators of gauge invariant (i.e. neutral) chiral fields

$$u = x_1^2 x_2, \quad v = x_3^2 x_2, \quad z = x_1 x_2 x_3. \tag{4.5}$$

These are not independent, and there is one relation among them:

$$uv = z^2. \tag{4.6}$$

So the geometry of the vacuum configuration (4.2) in this case is given by the A_1 ALE space. Turning on the FI D-term corresponds to blowing up the singularity. To see this, note that turning on the D-term corresponds to having the potential

$$V = (|x_1|^2 + |x_3|^2 - 2|x_2|^2 - A)^2, \tag{4.7}$$

where A corresponds to the FI term. If we take $A > 0$, x_1, x_3 cannot both be zero (in order to minimize V). The coordinates (x_1, x_3) up to an overall rescaling (which gets identified with the non-compact cotangent direction) can thus be viewed as a $\mathbf{P}^1$ whose Kähler class is controlled by A.

The geometrical interpretation of the above field theory makes it possible to use the powerful concept of toric geometry. This is not really necessary for the simple example above, but will be important for the more complicated cases, where the gauge theory picture becomes quickly unmanageable, whereas the toric methods proceed without trouble.

In toric geometry, the fields x_i become homogeneous variables on the quotient space $(\mathbf{C}^k - U)/(\mathbf{C}^\star)^r$, acted upon by the r $\mathbf{C}^\star$ actions (4.3). U is a subset of $\mathbf{C}^k$, defined by the $\mathbf{C}^\star$ actions and a chosen "triangulation" and we will determine it in a moment. It generalizes the point $x_i = 0$, $\forall i = 1, \dots, n+1$, that is removed in the case of ordinary projective space $\mathbf{P}^n$.

To find the gauge invariant fields we associate to each field x_i a vector $\nu_i = (\nu_{i,1}, \ldots, \nu_{i,d})$ in the standard lattice $\mathbf{Z}^d$, such that the ν_i fulfill the following relations determined by the $\mathbf{C}^\star$ actions

$$\sum_i l_i^{(a)} \nu_i = 0, \qquad \forall a = 1, \ldots, r \ . \tag{4.8}$$

Note that the dimension of the lattice is equal to the number of gauge invariant generators, $d = k - r$. Furthermore, for any vector $k_j \in \mathbf{Z}^d$,

$$u(k) = \prod_i x_i^{\langle \nu_i, k \rangle}, \qquad \langle \nu_i, k \rangle \equiv \sum_j \nu_{i,j} k_j,$$

is a gauge invariant field and we get therefore a convenient representations of the gauge invariant fields in terms of the integral vectors k_j.

Since only positive powers of the fields x_i should appear, we make the further restriction, that if N is the lattice generated by the vectors ν_i and if M is its dual lattice, the allowed choice for k lie in the cone in M defined by $\langle \nu_i, k \rangle \geq 0$. Moreover, to avoid redundancy of the description, we restrict to a set of generators $\nu_\alpha^\star \in M$ which generate all elements in this cone by positive coefficients. The generators of invariant fields are then $u_\alpha = \prod_i x_i^{\langle \nu_i, \nu_\alpha^\star \rangle}$.

Note that it follows from the anomaly freedom condition (4.1) that $\prod_i x_i$ is one of the invariant monomials. This in turn implies[4] the existence of a vector h with $\langle h, \nu_i \rangle = 1 \ \forall i$, that is the vertices ν_i lie in a hyperplane H of $\mathbf{Z}^d$. In geometrical terms this corresponds to the condition, that the singularities of the toric variety V are sufficiently well behaved to give rise to Calabi–Yau manifolds with first Chern class $c_1 = 0$. A convenient choice of coordinates is to take $h = (1, 0, 0, \ldots)$ and therefore $\nu_i = (1, *)$.

Applied to the above example defined by the charge vector (4.4) we get

$$\nu_i = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \nu_\alpha^\star = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \langle \nu_i, \nu_\alpha^\star \rangle = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and thus $u_\alpha \equiv u(\nu_\alpha^\star) = (u, z, v)$ as in eq. (4.5).

We still have to determine the disallowed set U. For this we will need some more technical definitions; however the final representation in terms of "toric diagrams" will be very transparent and is all what is needed to understand the following discussion.

The precise definition of the toric variety V is in terms of a collection of cones σ_μ, bounded by rays from the origin through the points in $\mathbf{Z}^d$ defined by the vertices ν_i. This is shown in fig. 2.

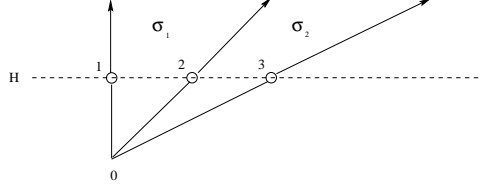---

[4] Neglecting subtleties related to torsion.

Fig. 2

The disallowed set U is now determined in the following way: Elements in U are defined by those subsets of vertices, which do not lie together in a single cone σ_μ:

$$\{x_i = 0, \ i \in \{i_\rho\}\} \in \ U \ \ \text{if} \ \ \{\nu_{i_\rho}\} \not\subset \sigma_\mu, \ \ \forall \sigma_\mu \ .$$

In the example above, $U = \{x_1 = x_3 = 0\}$, since the points ν_1, ν_3 do not lie in a cone (they are separated by the ray passing through ν_2).

From now on, we simplify the diagrams by suppressing the direction normal to H. The toric diagram for the above configuration looks than as in fig. 3.
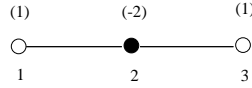

Fig. 3

A nice property of the toric variety V is that the hyperplanes $D_i : \{x_i = 0\}$, also called divisors, generate the $d - 1$ dimensional homology group and its dual, the homology class of curves we are after. In fact we are interested in the *compact* part of the toric variety and the curve classes C_a contained in it. The first rule to read the toric diagram can be stated as: the divisor $x_i = 0$ corresponding to the node ν_i is compact if ν_i is an interior node. In the example, the only interior node is ν_2, and $x_2 = 0$ can be readily seen to be compact, since the potential (4.7) implies $|x_1|^2 + |x_3|^2 = A$. Moreover the $\mathbf{P}^1$ with coordinates x_1, x_3 determined by this equation is the only homology class of the compact divisor D_2.

Physically, the most important quantities of the type IIA geometry are the intersections of the curve classes C_a contained in V. It is a standard calculation to determine these intersections from the relations (4.4) and U (for a pedagogical review see [36]). However in two complex dimensions there is a nice short-cut, which, in the presence of the fibration structure we use, will also be helpful for the threefold case and provides a direct link between the toric diagrams as in fig. 3 and Dynkin diagrams. Similar observations about the appearance of Dynkin diagrams in Calabi-Yau toric descriptions have been made in

[37] [38]. In fact there is the following simple way to read off the curve classes and their intersections: *each curve class C_a corresponds to a relation $l^{(a)}$ between the vertices ν_i. Moreover the entries $l_i^{(a)}$ are the intersections*[5] $C_a \cdot D_i$. In particular, in two complex dimensions, the hypersurfaces D_i are curves themselves and $C_a \cdot D_i$ is the intersection matrix for curves.

Note that the k divisors D_i are not independent but give rise to r different homology classes K_a, the Poincaré duals of the curves C_a. If we choose a preferred basis for the 2-cycles (thus fixing certain linear combinations of the $l^{(a)}$), namely such that the volumes of the curve classes C_a generate the Kähler cone of V, the intersections $C_a \cdot D_i$ reproduce precisely the Cartan matrix of the gauge system and part of the toric diagram agrees with the Dynkin diagram. The charge vectors $l^{(a)}$ in the basis dual to the Kähler cone are called Mori vectors.

In the $SU(2)$ example above, the Dynkin diagram of $SU(2)$ is given by the middle node ν_2 and we have also indicated the intersections of the single curve class $C = D_2$ with the non-compact divisors divisors D_1, D_3, (1), and with itself, (-2).

Now we consider the generalization of this to A_{n-1} ALE space. We consider a $U(1)^{n-1}$ theory with $n+1$ fields, with charges given by

$$l^{(1)} = (1, -2, 1, 0, 0, 0, \ldots, 0),$$
$$l^{(2)} = (0, 1, -2, 1, 0, 0, \ldots, 0),$$
$$l^{(3)} = (0, 0, 1, -2, 1, 0, \ldots, 0), \tag{4.9}$$
$$\vdots$$
$$l^{(n-1)} = (0, 0, 0, 0, \ldots, 1, -2, 1).$$

It is not too difficult to read off the geometry associated to this, just as in the A_1 case. In particular the generators of gauge invariant chiral fields are

$$u = x_1^n x_2^{n-1} x_3^{n-2} \ldots x_{n+1}^0,$$
$$v = x_1^0 x_2^1 x_3^2 \ldots x_{n+1}^n,$$
$$z = x_1 x_2 x_3 \ldots x_n,$$

with the single relation

$$uv = z^n, \tag{4.10}$$

---

[5] More precisely, it can happen that the intersections differ by a common normalization factor N_a. We take care of these factors in the following.

which defines the background geometry corresponding to A_{n-1} ALE space. Turning on the $n-1$ FI D-terms corresponds to blowing up the ALE space and introduces $n-1$ Kähler classes. Even though it is possible to study the linear sigma model phases and see the geometry of the resolved space, this becomes increasingly difficult. In fact it is precisely to answer such questions that toric geometry is useful. So let us see how this appears for the present example.

The toric data are now given by

$$
\nu_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & n \end{pmatrix}, \quad \nu_\alpha^\star = \begin{pmatrix} 1 & 0 \\ n & -1 \\ 0 & 1 \end{pmatrix}, \quad \langle \nu_i, \nu_\alpha^\star \rangle = \begin{pmatrix} 1 & n & 0 \\ 1 & n-1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 0 & n, \end{pmatrix} \tag{4.11}
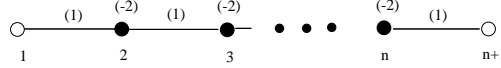$$

summarized in the toric diagram fig. 4

Fig. 4

The Dynkin diagram of $SU(n)$, associated to the curve classes C_a in the compact part of V, is plainly visible as the chain of black dots. We have also indicated the intersections of the 2-spheres contained in D_i, namely self-intersections (-2) and intersections (1) with the next neighbor. Note that they are given precisely by the entries of the charge vectors (4.9) and moreover agree with the entries of the Cartan matrix of $SU(n)$.

4.2. Local Mirror Symmetry

The data we used to describe the variety V, are the k vertices ν_i spanning a polyhedron Δ, the r relations $l^{(a)}$ fulfilled by them and the $k-r$ vectors $\nu_\alpha^\star$, that defined the generators of gauge invariant fields u_α. Note that the vertices $\nu_\alpha^\star$ define similarly a polyhedron $\bigtriangledown$ in the dual lattice. Batyrev's construction of the mirror geometry [39] proceeds by exchanging the roles of the polyhedra Δ and $\bigtriangledown$ (for an attempt to prove the mirror symmetry in terms of Batyrev's construction see [40]). More precisely, we consider a Calabi–Yau manifold X. In the simplest case, X is defined as a hypersurface in V, described by a homogeneous polynomial in the gauge invariant monomials u_α:

$$
0 = p(X) = \sum_\alpha b_\alpha u_\alpha \;=\; \sum_\alpha b_\alpha \prod_i x_i^{\langle \nu_i, \nu_\alpha^\star \rangle} \;=\; b_0\, x_1 x_2 \ldots x_k \;+\; \ldots ,
$$

22

where b_α are some complex numbers parameterizing the complex structure of X. Similarly the mirror polynomial, exchanging the roles of ν_i and $\nu_\alpha^\star$ is given by

$$0 = p(X^\star) = \sum_i a_i y_i \;=\; \sum_i a_i \prod_\alpha \tilde{x}_\alpha^{\langle \tilde{\nu}_\alpha^\star, \nu_i \rangle} \;=\; a_0\, \tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_k \;+\; \ldots \;, \tag{4.12}$$

where $\tilde{x}_\alpha$ are the homogeneous coordinates of $V^\star$, $\tilde{\nu}_\alpha^\star$ are vertices in the convex hull spanned by the vertices $\nu_\alpha^\star$ and a_i parameterize the complex structure of $X^\star$. Moreover y_i are monomials in the variables $\tilde{x}_i$ invariant under the $\mathbf{C}^\star$ actions $\tilde{x}_\alpha \to \tilde{x}_\alpha \lambda^{\tilde{q}_\alpha^a}$, which descend from relations $\sum \tilde{l}_\alpha^{(a)} \tilde{\nu}_\alpha^\star$ fulfilled by the dual vertices $\tilde{\nu}_\alpha^\star$. The statement of mirror symmetry is that the geometry of Kähler variations of the space X is captured by the complex deformations of the space defined by (4.12). Moreover there is a precise procedure to read off the instanton corrections of the original space, in terms of "variations of hodge structure" of the mirror geometry given by (4.12) (in terms of specific period integrals).

The charge vectors $l^{(a)}$ of X imply the following relations[6] between the gauge invariant coordinates y_i of $X^\star$:

$$\prod_i y_i^{q_i^a} = \prod_\alpha \tilde{x}_\alpha^{\langle \tilde{\nu}_\alpha^\star, \sum_i l_i^{(a)} \nu_i \rangle} = 1 \qquad \forall a \tag{4.13}$$

Note that these equations can be studied for sets of relations $l^{(a)}$ independently of an embedding in a larger system. In particular, as in [12], consider the hyperplane given by

$$p(X^\star) = \sum a_i y_i = 0, \tag{4.14}$$

where i runs only over the vertices ν_i describing the local geometry of the gauge system. This equation is homogeneous, and one can eliminate an overall scale from (4.14) and so we end up with $k - r - 2$ dimensional space as the mirror. This can be two lower than the dimension expected in generic applications in the compact cases as was in the A_{n-1} case above.

This reduction of dimension (and the generalization to the complete intersection case) can be understood in the following way. Suppose we start with a (possibly non-compact) Calabi–Yau X described by $\hat{k}$ vertices $\hat{\nu}_i$ and $\hat{r}$ relations $\hat{l}^{(a)}$. The dimension of X would be $\hat{d} = \hat{k} - \hat{r}$ without a superpotential and $\hat{d} = \hat{k} - \hat{r} - 2$ with a superpotential $p(X)$, one

---

[6] The relation (4.6) fulfilled by the gauge invariant fields of the original manifold X is a relation of this kind associated to the Mori generators of $X^\star$, $\tilde{l}_\alpha^{(a)}$.

less from the equation and one less because of (4.1). Now divide the vertices $\hat{\nu}_i$ into two sets, the first, $\Delta^0 = \{\nu_i\}$, containing the k vertices describing the local geometry of the gauge system and the second one, $\Delta^0{}' = \{\nu_i'\}$ containing the rest. Similarly we divide the relations $\hat{l}^{(a)}$ into two sets, according to whether or not they involve elements of $\{\nu_i\}$. Let r be the number of relations $l^{(a)}$ involving some of the $\{\nu_i\}$.

Two situations can arise: i) the d dimensional local geometry describing the gauge system is constrained, that is the singularity exists only on the hypersurface $p(X) = 0$. In this case, $d = k - r - 2$ and the mirror geometry is of the same dimension. This happens e.g. for a D_n singularity discussed in a later section. ii) the d dimensional local geometry describing the gauge system is unconstrained. In this case we have $d = k - r$, a case without a superpotential. However the mirror geometry *is* constrained by $p(X^\star)$, giving a mirror of dimension $d = k - r - 2 = d - 2$. In particular, this happens for A_n singularities where one obtains Riemann surfaces as the mirror geometry of a threefold. In such cases, one can relate the type IIB theory to a d-fold geometry by noting that adding two more variables to the equation which appear quadratically, $p(X^\star) = 0 \to p(X^\star) + uv = 0$ does not affect the period integrals and so we can view the mirror geometry as this local d-fold (the trick of adding quadratic variables to describe the geometry is familiar from the study of LG models [41] [42]).

Sometimes simplifications can occur in describing the mirror. In particular if there are variables which appear only linearly, they serve as 'auxiliary' fields and can be eliminated by setting to zero the variation of the polynomial with respect to them, without affecting the period integrals of the mirror. In particular if we have several variables x_0^i which appear linearly in the polynomial[7], $p(X^\star) = \sum x_0^i G_i$, the mirror geometry can be viewed as corresponding to the complete intersection $G_i = 0 \quad \forall i$.

Let us see how this mirror symmetry works in the case of A_{n-1} which we have constructed above. In this case we have $y_1, ..., y_{n+1}$ as the space of $y's$ subject to the relation (4.13)

$$y_i y_{i+2} = y_{i+1}^2.$$

Choosing the homogeneous factor so that $y_1 \to 1$ we can solve the above relations[8] and obtain

$$(y_1, y_2, \ldots, y_{n+1}) = (1, y, y^2, \ldots, y^n),$$

---

[7] Note that the existence of the hyperplane H implies that there is always one variable which appears only linearly, related to the hypersurface constraint (4.12).

[8] Or simply using $y_i = \prod \tilde{x}_\alpha^{\langle \nu_i, \nu_\alpha^\star \rangle}$ with $\langle \nu_i, \nu_\alpha^\star \rangle$ given in (4.11) and $\tilde{x}_\alpha = (x_0, y, s \equiv 1)$.

where we have defined $y = y_2$. And thus the mirror geometry is

$$\sum_{i=1}^{n} a_i y^i = uv$$

where we have introduced the auxiliary variables u, v to make contact with 2-fold geometry. The result is as expected, namely the Kähler deformations of the A_{n-1} singularity has been changed to deformation of the same singularity (the self-mirror property of A_{n-1} ALE space already discussed in section 3).

4.3. The Threefold Case

The type IIA compactification on the A_{n-1} geometry described in the previous section develops an enhanced $SU(n)$ gauge symmetry in six dimensions. As discussed previously, to get an $N = 2$ theory in four dimensions we have to compactify further on a one complex dimensional space which should have no 1-cycles to avoid adjoint matter - that is a collection of 2-spheres. Therefore we have to consider *base geometries that are precisely of the same type as the fiber geometries.*

This makes the discussion of the threefold geometry simple. As before let us start with the simplest case A_1, a single 2-sphere. However we have now two such $\mathbf{P}^1$'s, one for the fiber and one for the base. Moreover, instead of taking only the naive product of the two $\mathbf{P}^1$'s, we can take instead non-trivial fibration of the first $\mathbf{P}^1$ over the second one (see [12] for a detailed treatment of the mirror of such cases that we review below).

These $\mathbf{P}^1$ bundles over $\mathbf{P}^1$ are classified by a single integer n and are called Hirzebruch surfaces, denoted $\mathbf{F}_n$. Let us describe them in the notation we introduced in the previous section. The geometry of the A_1 singularity with a single blow up sphere we considered in detail, was conveniently summarized in the toric diagram fig. 3. For two $\mathbf{P}^1$'s, one for the fiber and one for the base, we will have to combine two of these geometries. The only non-trivial question is how they are connected, in other words, to specify the fibration. Let us continue to reserve the horizontal direction in the diagrams to denote the fiber geometry whereas we use now the vertical direction to draw the geometry for the base. The result is shown in fig 5.
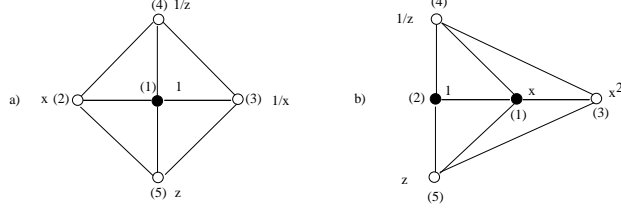
Fig. 5: Local threefold geometries: a) $\mathbf{F}_0 : \mathbf{P}^1 \times \mathbf{P}^1$ b) $\mathbf{F}_2$.

Note that the fiber (base) $\mathbf{P}^1$ corresponds to simply omitting the points ν_4, ν_5 (ν_2, ν_3). Not surprisingly, the two geometries in fig. 5 differ in the fibration, the first one corresponding to the trivial product $\mathbf{F}_0 : \mathbf{P}^1 \times \mathbf{P}^1$ whereas the second geometry has a non-trivial fibration and is an $\mathbf{F}_2$ surface. To see this, let us write down the vertices and charge vectors:

$$
\mathbf{F}_0 : \nu_i = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad l^{(a)} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix},
$$

$$
\mathbf{F}_2 : \nu_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad l^{(a)} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{pmatrix}.
$$

(4.15)

Recalling that the charge vectors $l_i^{(a)} = q_i^a$ define the $\mathbf{C}^\star$ actions $x_i \to \lambda^{q_i^a} x_i$, we see that in the first case the projective actions of the $\mathbf{P}^1$ factors are independent, $(x_1, x_2, x_3, x_4, x_5) \to (\lambda^{-2}\mu^{-2}x_1, \lambda x_2, \lambda x_3, \mu x_4, \mu x_5)$, whereas in the second case the coordinates of the fiber $\mathbf{P}^1$, namely (x_2, x_3), transform non-trivially under rescalings of the coordinates (x_4, x_5) of the base $\mathbf{P}^1$, $(x_1, x_2, x_3, x_4, x_5) \to (\lambda^{-2}x_1, \lambda x_2, \lambda\mu^{-2}x_3, \mu x_4, \mu x_5)$. We have also indicated in fig. 5 the variables that solve (4.13) and appear in the hypersurface constraint $p(X^\star)$.

Note that the first coordinate, x_1, is necessary to satisfy the anomaly cancellation (4.1). It corresponds to the non-trivial canonical bundle of the Hirzebruch surfaces $\mathbf{F}_n$. To ensure the Calabi–Yau condition $c_1 = 0$ we have to consider the 3 complex dimensional total space.

In the large base limit, which is the relevant one for the weakly coupled field theory limit (see (2.2)), the difference between the two fibrations $\mathbf{F}_0$ and $\mathbf{F}_2$ is actually irrelevant. It is only the stringy strong coupling behavior, corresponding to small base, in which they differ. We will discuss these points as we go along with the solution of the more general theories.

26

Let us finally sketch the appearance of matter. As explained already, matter arises from extra singularities, localized above special points on the base geometry. Geometrically this corresponds to introducing extra $\mathbf{P}^1$'s, blowing up points on the base. This is shown for the $SU(3)$ gauge theory with $N_f = 1$ matter in fig. 6 a).
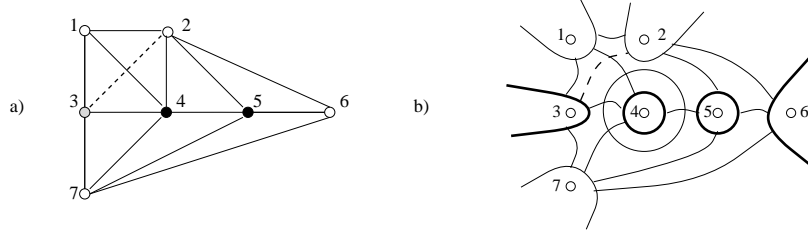

Fig. 6: Geometry for the $SU(2)$ $N_f = 1$ theory: a) toric polyhedron Δ of the type IIA geometry b) Riemann surface of the type IIB geometry.

As in the previous example, the base is a simple $\mathbf{P}^1$ factor represented by the three points ν_1, ν_3, ν_7 on the vertical line. The horizontal line with points $\nu_3, \nu_4, \nu_5, \nu_6$ describes the case $n = 3$ of eq. (4.11), an A_2 singularity. Inspection of the relations $l^{(a)}$ as in (4.15) identifies the compact divisors ν_4 and ν_5 as a blowup of $\mathbf{F}_1$ and the $\mathbf{P}^1$ bundle $\mathbf{F}_3$, respectively. The blow up $\mathbf{P}^1$ corresponding to the matter corresponds to the extra point ν_2.

The above geometry is also a simple example where it is possible to choose two different partitions into cones, denoted by the dashed line in the figure. In particular this means that the specification of the vertices alone does not determine the geometry completely. Drawing rays through the points ν_i, as in fig. 2, is not sufficient to generate a valid collection of cones, which is characterized by the property that the projection of the faces to the hyperplane H we draw, should yield a triangulation of the polyhedron Δ defined by the points ν_i.

Physically, a choice of triangulation corresponds to the fact that the spectrum of light relevant BPS states can depend on the region in moduli space one considers. A simple representation of this fact is shown in fig. 6b) which displays the Riemann surface E representing the mirror geometry of the toric geometry in fig. 6a). Each interior point of the polyhedron Δ corresponds to a non-trivial homology class of E and moreover each link of Δ to a non-trivial 1-cycle on E. Moduli of the gauge theory are associated with periods along 1-cycles in the compact part of E whereas bare parameters as the gauge couplings mass parameters arise from 1-cycles that wrap the non-compact legs of E, related to the behavior "at infinity". Depending on the region in moduli space, the period over the

27

1-cycles between the points ν_1 and ν_4 may be smaller or bigger than the one over the dashed cycle between the points ν_2 and ν_3, Choosing the one that leads to the smaller value corresponds to choosing the triangulation of Δ.

4.4. Mirror Map on Moduli Space and the Exact Solution

Let us finally collect the necessary ingredients to determine the exact solutions for the moduli dependent gauge couplings from the geometry.

In the type IIA compactification, the volumes V_a of the 2-spheres combine together with the anti-symmetric tensor fields B_a to form complex fields t_a parameterizing the Kähler moduli space. In the Mori basis we introduced previously, the Kähler moduli t_a correspond 1-1 to the charge vectors $l^{(a)}$, as is clear from their interpretation as FI parameters in the linear sigma model. Mirror symmetry, now on the moduli space, relates the Kähler moduli t_a of the type IIA geometry X to complex structure moduli z_a of the mirror geometry $X^\star$

$$t_a = B_a + iV_a = \frac{1}{2\pi i} \ln z_a + \mathcal{O}(z_a),$$

where the so-called algebraic coordinates z_a are given by

$$z_a = \pm \prod_{i=1}^{r} a_i^{l_i^{(a)}} \tag{4.16}$$

with the a_i defined as in (4.12). The exact worldsheet instanton corrected prepotential is then obtained from the period integrals of the unique holomorphic $(d, 0)$ form Ω on $X^\star$, in the general form given in [39]. Another expression which turns out to be useful in certain cases, is the logarithmic form:

$$\Omega = \ln(p(X^\star)) \prod_{i=1}^{m} \frac{dy_i}{y_i} \ . \tag{4.17}$$

It is straightforward to check, that the period integrals $\Pi_i = \int_{\gamma_i} \Omega$ of Ω, where γ_i is a basis of non-trivial homology d-cycles, fulfill the GKZ system of differential equations

$$\prod_{l_k^{(i)} > 0} \left(\frac{\partial}{\partial a_i}\right)^{l_k^{(i)}} = \prod_{l_k^{(i)} < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_k^{(i)}} \ . \tag{4.18}$$

Moreover, as a consequence of the $\mathbf{C}^\star$ actions, the period integrals depend only on the combinations z_i of the a_i defined in (4.16). This invariance can be similarly expressed

28

in terms of differential equations. They depend in addition on a choice of gauge for Ω which fixes the proportionality factor that depends only on the complex parameters a_i. The Calabi–Yau periods are then obtained by choosing linear combination of the solutions to (4.18) with leading behavior determined by the intersections. Specifically, the perturbative one-loop prepotential of the four-dimensional $N = 2$ gauge theory, defined by a choice of a root system R_G corresponding to a gauge group G and matter representations corresponding to weight vectors $w \in W_G$, is given by

$$\mathcal{F} = \frac{i}{4\pi} \sum_{\alpha \in R_G} (a \cdot \alpha)^2 \ln \frac{(a \cdot \alpha)^2}{\Lambda^2} - \frac{i}{4\pi} \sum_{w \in W_G} (a \cdot w)^2 \ln \frac{(a \cdot w)^2}{\Lambda^2}, \qquad (4.19)$$

where a_i are the Coulomb moduli parameterizing the moduli space of the $N = 2$ theory. From the string point of view, the perturbative couplings (4.19) describe the intersections of the type IIA geometry X. Once we have completed the geometrical engineering of the appropriate local geometry of 2-spheres, such that the intersections reproduce the perturbative piece (4.19), the exact solution is immediately determined in terms of the Picard-Fuchs system (4.18)[9]. Note the amazingly direct relation between the group theoretical data ($\alpha \in R_G, w \in W_G$) defining the perturbative field theory, and its exact geometrical solution in terms of the differential equations (4.18). *Essentially all we need is the direct correspondence between the charge vectors $l^{(a)}$ and (α, w) defining the 2d and 4d field theories, respectively.* This is similar to statements about the relation of Picard-Fuchs equations with the matter representations made in the context of $N = 2$ field theories in [45][46].

The periods Π_i describe the exact special geometry of the Calabi–Yau moduli space. To get the rigid special geometry of the field theory moduli space we have still to decouple gravity effects, as in [5]. We will describe the appropriate limit when we treat the general case of a product gauge group in section sect 5.2.

5. Linear Chain of $\prod_i SU(k_i)$ with bi-Fundamental Matter

We will now describe the geometrical construction of the exact solution of the linear chain of SU groups with bi-fundamental matter between adjacent groups. We begin with a simple case of $SU(N+1)$, $N_f = 1$ and gradually add the different building blocks needed to describe the most general case.

---

[9] For details on Calabi–Yau techniques to determine the periods from the GKZ system, see [43] [44].

5.1. $SU(N+1)$ with $N_f = 1$

We start with the construction and solution of the model shown in fig. 6, with the generalization that we consider general $SU(N+1)$ instead of $SU(3)$. The type IIA geometry X is therefore defined by the $n_\nu = N + 5$ vertices ν_i in $\mathbf{R^2}$:

$$\nu_{1,0} = (1,0), \ \nu_{2,\mathrm{i}} = (2,\mathrm{i}), \ \nu_{3,\mathrm{k}} = (3,\mathrm{k}) \ , \tag{5.1}$$

with $i \in \{0, \dots, N+1\}$ and $k \in \{0, 1\}$. To make contact with the gauge theory we identify the classes C_i of the 2-cycles and their intersections. As explained previously, the classes C_i are in 1-1 correspondence with linear relations of the vertices ν_i, described by the charge vectors

$$\begin{aligned}
\mathrm{l}^{(g)} &= (1, -2, 0^{N+1}, 1, 0), \\
l_i^{(c)} &= (0^i, 1, -2, 1, 0^{N-i}, 0^2), \ i = 1, \dots, N \ , \\
l^{(m)} &= (0, -1, 1, 0^N, 1, -1) \ ,
\end{aligned} \tag{5.2}$$

where we introduce the following notation: parameters related to bare gauge couplings will be denoted by a superscript (g), and similarly bare masses by (m) and Coulomb parameters by (c).

Local Type IIA Geometry

Let us describe the local geometry of the above toric variety X in some more detail; in particular we want to show that it contains the homology of 2-cycles with the appropriate intersections. The derivation in the following paragraph requires some more knowledge of toric geometry which is not needed otherwise to follow the remaining discussion.

The polyhedron Δ defined by (5.1) describes a toric variety, whose compact part is composed of a chain of N rational ruled surfaces E_i, $i = 1, \dots, N$. In the above model, the E_i for $i > 1$ are Hirzebruch surfaces $\mathbf{F}_{n_i}$ with $n_i = 2i - 1$, and the first one, E_1, is a blow up of $\mathbf{F}_2$ along the intersection of the section and the fiber. The blowup corresponds to the additional vertex $\nu_{3,1}$. The curve classes can be described as follows: each $\mathbf{F}_{n_i}$ has two sections s_i, t_i with intersections $s_i^2 = -n_i$, $t_i^2 = n_i$, $s_i \cdot t_i = 0$ and a fiber f_i class with $f_i \cdot s_i = 1$. Since the compact divisors E_i meet along sections, there is only one independent class from the sections, which we can choose to be the section of the first factor, E_1. In addition we have N fibers f_i and the exceptional curve of the blow up, u, which supports

the matter hypermultiplet. In summary we have $N + 2$ curve classes s_1, f_i, u with the following intersections with the divisors E_i:

$$s_i \cdot E_i = \begin{cases} -2 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}, \quad f_i \cdot E_j = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases},$$

$$u \cdot E_i = \begin{cases} -1 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

where we denoted by E_0 the non-compact divisor at one end of the A_N chain. Note that these intersections are precisely the entries of the Mori vectors (5.2). Moreover, $l^{(g)}$ and $l^{(m)}$ describe the charges of the $SU(N + 1)$ vector bosons and the highest weight of the fundamental matter in a Cartan basis, respectively.

Type IIB Mirror Geometry:

The intersections (5.3) guarantee the correct leading behavior described by the perturbative gauge couplings (4.19). To solve for the exact dependence including instanton corrections, we apply now the local mirror transformation (4.13) to the above type IIA geometry parametrized by the Kähler moduli to obtain the type IIB geometry parametrized by complex deformations. A solution is given by the monomials

$$\frac{1}{z}, s^{N+1}, s^N w, \dots, w^{N+1}, z s^{2N+2}, z w s^{2N+1} .$$

and after setting $s = 1$, using the $\mathbf{C}^\star$ action $(z, w, s) \to (\lambda^{-N-1} z, \lambda w, \lambda s)$, we obtain the defining equation for the mirror manifold $X^\star$, a Riemann surface:

$$p(X^\star) = \frac{a_{1,0}}{z} + P_{N+1}(w) + z(a_{3,0} + a_{3,1} w) , \quad P_{N+1}(w) = \sum_{n=0}^{N+1} a_{2,n} w^n . \tag{5.4}$$

The moduli $a_{i,j}$ are related to the vertex $\nu_{i,j}$ carrying the same index. Note that although the curve (5.4) is appropriate to describe the field theory, the expression (4.17) for the holomorphic 1-form, $\Omega = \ln(w) dz/z$ is not (yet). Only after taking the large base space limit, which requires a shift of the w variable to stay near the singularity, the answer will agree with the field theory answer. We discuss the precise limit together with the more general case in the next section.

5.2. The Linear Chain with bi-Fundamentals

Let us consider now the general case of products of $SU(N)$ factors with matter in the bi-fundamental representations. In the case of a single $SU(N)$ factor, we had a single $\mathbf{P}^1$ as the base geometry. For the generalization to a product of $SU(N)$ factors we have simply to replace the single $\mathbf{P}^1$ by an A_M chain of $\mathbf{P}^1$'s with nearest neighbors intersecting. The toric diagram now looks like shown in fig. 7.
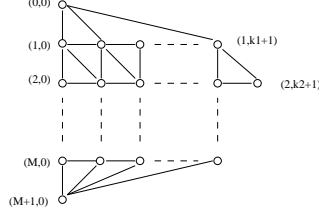
Fig. 7: Polyhedron Δ for the product gauge group $\prod_{r=1}^{M} SU(k_i + 1)$.

The vertical line at the left end describes an A_M Dynkin diagram associated with the base geometry, whereas the horizontal lines describe the SU factors from the fibers.

In fig. 7 we have implicitly assumed the convexity of the polyhedron Δ. The convexity assumption is needed for the validity of the mirror description in terms of polyhedra. This can be easily seen to be equivalent to the appropriate high-energy behavior of the product gauge theory: The matter content of a given $SU(k_r + 1)$ factor is $(k_{r-1} + 1) + (k_{r+1} + 1)$ fundamental representations. Asymptotic freedom requires $2k_r \geq k_{r-1} + k_{r+1}$, in agreement with the convexity of the polyhedron Δ.

However also the non-convex toric diagrams have a valid physical interpretation. Assume we start from a convex toric diagram as in fig. 7, but tune the moduli of some of the vertices on the right side towards zero. This is an allowed moduli and leads to some $\mathbf{P}^1$'s becoming large. Therefore the corresponding gauge bosons and matter fields associated with D2 branes wrappings get very heavy. At low energies, the theory is described effectively by states associated with the small 2-cycles, which might well be described by the homology classes of a non-convex toric diagram. However we should think of such a diagram as being part of a more complicated theory, including the vertices required by convexity.

There are two types of situations which may arise; the toric diagrams are depicted in fig. 8. In the first case, completing the diagram effectively enlarges the rank of the previously infrared free SU factor. In this case the consistent high energy behavior is restored by the presence of additional charged vector bosons and their negative contribution to the beta function. In the second, more interesting case, the rank of the gauge group stays the same. The consistent high energy behavior arises from a non-trivial coupling of the infrared free SU factor to the other group factors.
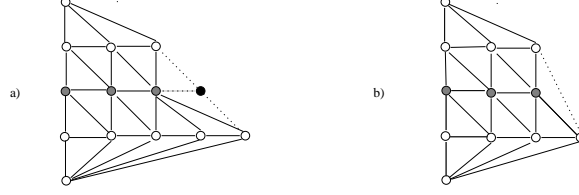
Fig. 8: Toric completion of an infrared free gauge group factor (grey vertices) a) convexity requires an additional *vertex* ν' (black circle), b) the completion of the convex polyhedron requires only a new *link* between the neighbored SU factors (dotted line).

The vertices $\nu_i \in \mathbf{R^2}$ of the polyhedron Δ can be read off from fig. 7. For a product gauge group $\prod_{r=1}^{M} SU(k_r + 1)$, we have $n_\nu = 2 + 2M + k$ vertices, where $k = \sum_r k_r$. There are $n_{C_i} = 2M - 1 + k$ independent classes of 2-spheres C_i, whose volumes describe the k Coulomb parameters $z_{r,i}^{(c)}$, $M - 1$ bare masses $z_s^{(m)}$ and M coupling constants $z_i^{(c)}$:

$$
\begin{aligned}
z_{r,i}^{(c)} &= \frac{a_{r,i-1}a_{r,i+1}}{a_{r,i}^2} \quad,\quad r = 1,\ldots M \qquad\qquad , \ i = 1,\ldots,k_r \ , \\
z_r^{(m)} &= \frac{a_{r,0}a_{r+1,1}}{a_{r+1,0}a_{r,1}} \quad,\quad r = 1,\ldots,M-1 \ , \\
z_r^{(g)} &= \frac{a_{r-1,0}a_{r+1,0}}{a_{r,0}^2} \quad,\quad r = 1\ldots M \ .
\end{aligned}
\tag{5.5}
$$

The naive dimension of the mirror manifold is $n_\nu - n_{C_i} = 3$. However this is a case where we had no superpotential on the type IIA side and we expect a reduction of dimension by two as explained previously. The Mori vectors $l^{(i)}$ can be easily read off from equations (5.5), (4.16). The local mirror geometry, given as the solution of (4.13) with the Mori vectors defined as above, describes a Riemann surface $X^\star$ given as a hypersurface:

$$
p(X^\star) = a_{0,0} + \sum_{r=1}^{M} z^r P_r(w) + a_{M+1,0} z^{M+1}, \qquad P_r(w) = \sum_{l=0}^{k_r+1} a_{r,l} w^l.
$$

To reduce to the field theory limit, we shift w by a constant $\sim \epsilon^{-1}$ and send $\epsilon \to 0$. This shift has to be accompanied by a corresponding limit in moduli space to stay near the singular point of the A_{k_r} singularity described by $P_r(w)$. This identifies

$$
a_{r,l} \sim \epsilon^{l-k_r-1}
\tag{5.6}
$$

as the correct limit. Furthermore $a_0, a_{M+1} \sim \epsilon^0$. The algebraic coordinates (5.5) scale in this limit as as

$$
z_r^{(c)} \sim z_r^{(m)} \sim \epsilon^0, \qquad z_r^{(g)} \sim \epsilon^{-b_r} \ ,
$$

33

where $b_r = k_{r-1} + k_{r+1} - 2k_r$ is the beta-function coefficient of the r-th group factor. For asymptotic free theories $b_r < 0$ and the limit $\epsilon \to 0$ corresponds to taking a very large base, as expected: From (4.16) the volume of the r-th 2-sphere in the base scales as $V_r \sim b_r \ln \epsilon$. Moreover, for each configuration saturating the limit, $2k_{r+1} = k_r + k_{r+2}$, we gain a new finite parameter $z_{r+1}^{(g)}$ in the field theory limit, corresponding to an undetermined bare coupling τ_r. Note that this parameter appears naturally as the Coulomb modulus of a SU factor from the base. This will be important when we identify the S-duality group and its physical origin.

We complete the discussion of the chain model with an explanation of the local type IIA geometry. Similarly as in the case of a single $SU(N)$ factor, the interior points of the polyhedron describe the compact divisors E_r^i of X. We have $i = 1, \ldots, k_r$ of such exceptional divisors, ruled over the r-th base $\mathbf{P}_r^1$, which itself intersects transversally the adjacent spheres $\mathbf{P}_{r-1}^1$ and $\mathbf{P}_{r+1}^1$ in the base. The divisors E_r^i are Hirzebruch surfaces $\mathbf{F}_{n_r^i}$ blown up along the intersection of t_r^i with a fiber (fiber over the same point of $\mathbf{P}^1$ for all i). The proper transforms are denoted by $\widetilde{t_r^i}$. There are also the exceptional curves u_r^i introduced by the blowups.

E_r^i and E_r^{i+1} are glued together by attaching the sections $\widetilde{t_r^i}$ to s_r^{i+1}. Since the normal bundles of this common curve in E_r^i and E_r^{i+1} can be seen to add up to -2, this can be embedded locally in a Calabi-Yau threefold.

Finally E_r^i meets E_{r+1}^i by the identification of u_r^i with u_{r+1}^i; these are each curves with self-intersection -1, so the union can be locally embedded in a Calabi-Yau threefold. Note that we have now one fiber with $k + M - 1$ components. The Mori cone is generated by these $k + M - 1$ components of the fiber, together with M sections s_r, giving the total of $k + 2M - 1$ generators.

6. Trivalent Geometry and Addition of Fundamental Matter

To add N_f matter in the fundamental representation for a given gauge group $SU(N_c)$, we can introduce an extra sphere which intersects a given $\mathbf{P}^1$ base corresponding to the gauge group of interest, on top of which we have an A_{N_f-1} singularity. In this way we obtain extra bi-fundamental matter (N_f, N_c). By degenerating the extra sphere, i.e. by considering the extra sphere to be very large, we weaken the $SU(N_f)$ dynamics, thus 'demoting' it to a spectator flavor symmetry group. In this section we would like to add matter to the linear chain of SU groups considered in the previous section. In order to

construct this geometry along the lines just discussed we clearly need an additional building block of our base geometry, corresponding to the trivalent vertices in the base. That is we need a central sphere of self-intersection -2 intersecting three other 2-spheres once.

A non-trivial constraint arises from the fact that this geometry has to be compatible with the Calabi–Yau condition. Recall that each Mori vector defines a 2-cycle C_i which is a generator of the canonical base for the 2-cycle homology and moreover that the intersections of this 2-cycle with the divisors in X are proportional to the entries of the Mori generator. If X is Calabi–Yau, the entries of each Mori generator always have to add up to zero[10]. The Calabi–Yau geometry forces us therefore to introduce one other vertex with contribution -1 such that the Mori vector of the central 2-sphere becomes $l^{(0)} = (-2, 1, 1, 1, -1, 0, \ldots)$. The local mirror geometry, solving (4.13), contains now the monomials

$$1, x, y, z, xyz \ , \tag{6.1}$$

where we have used a $\mathbf{C}^\star$ action of the solution to scale one of the variables to 1.

The solution to the local mirror geometry of the trivalent vertex uses two more variables than the 2-vertex we used previously in the chain. If we want to keep the dimension of the manifold X three, adding trivalent vertices, we have to add more equations, that is, we have to consider a complete intersection manifold.

6.1. Gauge Group $SU(N)$ with M Fundamental Matter Revisited

Before proceeding to the case of linear chain with extra fundamental matter, let us consider the special case of $SU(N)$ with M fundamental matter. We have already done this by viewing it as a special case of the chain of $SU(N) \times SU(M)$ where the $SU(M)$ coupling is weakened. Now we wish to obtain it again using the trivalent geometry constructed above. The type IIA geometry is shown in fig. 9a).

---

[10] The fact that $c_1 = 0$ implies $\sum_i l_i^{(a)} = 0$ is a basic result in toric geometry. Alternatively, in the linear sigma model language, $\sum_i l_i^{(a)}$ describes the anomaly contribution to the a-th $U(1)$ factor (4.1).
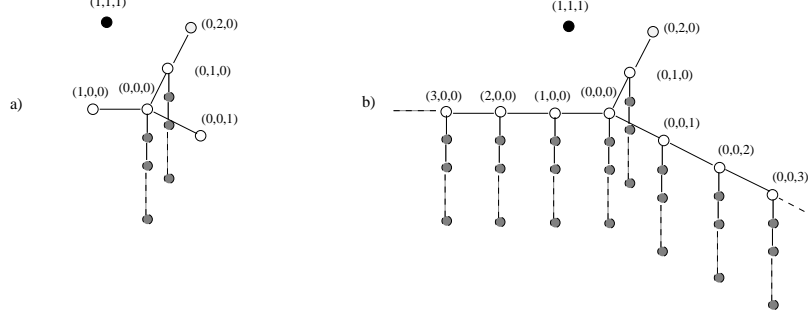
Fig. 9: a) SU gauge group with fundamental matter from the trivalent vertex geometry, b) Product gauge group $\prod SU(l_r) \cdot SU(n) \cdot \prod SU(k_r)$ with bi-fundamentals and fundamental matter in the $SU(n)$ factor.

The vertices for the base geometry can be read off from the figure (we add an extra zero at the end of each vertex for the fiber direction). The fiber geometry involves an A_{N-1} chain of $N - 1$ 2-spheres fibered over $s_{0,0,0}$, the central base $\mathbf{P}^1$, generating the gauge symmetry, and a further A_{M-1} chain of $M - 1$ 2-spheres fibered over another base $\mathbf{P}^1$, $s_{0,1,0}$ (in an obvious notation), which generates the matter. These two chains are described by two sets of vertices $\nu_{\mathrm{k}}^{0,0,0} = (0,0,0,\mathrm{k})$, $\mathrm{k} = 1, \ldots, \mathrm{N}$ and $\nu_{\mathrm{k}}^{0,1,0} = (0,1,0,\mathrm{k})$, $\mathrm{k} = 1, \ldots, \mathrm{M}$, respectively. By sending the gauge coupling of the $SU(M)$ gauge group to zero, the "degeneration" process described previously, we are left only with $SU(N)$ gauge group with M fundamentals. Geometrically we take the base $\mathbf{P}^1$ of the A_{M-1} factor to have infinite size[11]. In the toric geometry, this is easily done by simply cutting the length of the leg carrying the A_{M-1} chain to one, that is deleting the vertex $\nu^{0,2,0}$ from our polyhedron.

We proceed with the solution of the model. The $(N) + (M) + (4 + 1)$ vertices fulfill $N + M$ relations corresponding to the $N - 1$ Coulomb parameter $z_i^{(c)}$ of $SU(N)$, M mass parameters $z_i^{(m)}$ and the universal scale parameter $z^{(g)}$:

$$z_{\mathrm{i}}^{(c)} = \frac{a_{\mathrm{i}-1}^{0,0,0} a_{\mathrm{i}+1}^{0,0,0}}{(a_{\mathrm{i}}^{0,0,0})^2}, \quad z_0^{(\mathrm{m})} = \frac{a_0^{0,0,0} a_1^{0,1,0}}{a_1^{0,0,0} a_0^{0,1,0}}, \quad z_{\mathrm{i}}^{(\mathrm{m})} = \frac{a_{\mathrm{i}-1}^{0,1,0} a_{\mathrm{i}+1}^{0,1,0}}{(a_{\mathrm{i}}^{0,1,0})^2}, \quad z^{(g)} = \frac{a_0^{0,1,0} a_0^{1,0,0} a_0^{0,0,1}}{(a_0^{0,0,0})^2 a_0^{1,1,1}}. \tag{6.2}$$

The mirror geometry $X^\star$ is described by the hypersurface

$$p(X^\star) = P_N(w) + x + y + z \left(a_0^{1,1,1} \, xy + Q_M(w)\right),$$

$$P_N(w) = \sum_{i=0}^{N} a_i^{0,0,0} \, w^i, \qquad Q_M(w) = \sum_{i=0}^{M} a_i^{0,1,0} \, w^i , \tag{6.3}$$

---

[11] Note that the 2-cycles supporting the matter multiplets sit at a common point of this $\mathbf{P}^1$. Intuitively the limit corresponds to restricting to the neighborhood of this point on the base $\mathbf{P}^1$ while deleting the dependence on the global structure - this will effectively lead to a reduction of the dimension of X.

where we have set various coefficients to one using the $\mathbf{C}^\star$ symmetries of the solution. Note that z appears only linearly in $p(X^\star)$ and can be integrated out. This integration imposes the constraint

$$e = xy + Q_M(w) = 0 \ ,$$

and solving $e = 0$ for, say, y, we get finally

$$p(X^\star) = x + P_N(w) + \frac{Q_M(w)}{x} \ ,$$

which is the expected result.

6.2. Product Gauge Group $\prod SU(k_r)$ with $(k_r, \bar{k}_{r+1}) \oplus m_r \cdot (k_r)$ matter

The above construction generalizes straightforwardly to the product gauge groups considered previously. For each single group factor $SU(k_r)$ we replace the base $\mathbf{P}^1$ by the central $\mathbf{P}^1$ of the trivalent geometry, as shown in fig. 9b) for one SU factor.

The base geometry is now a chain of T trivalent vertices, where above each central $\mathbf{P}^1$ we have an A_{k_r-1} chain of 2-spheres generating the gauge symmetry and above each layer $\mathbf{P}^1$ there is an A_{m_r-1} chain of spheres adding the fundamental matter. The vertices of the toric type IIA geometry can be obtained from plumbing together those of a single trivalent vertex as described before. They can be also recovered from our description of the type IIB mirror geometry below.
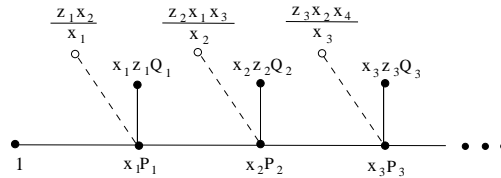

Fig. 10

Fig. 10 shows the structure of our chain of trivalent vertices, where we have indicated the monomials that solve the mirror geometry. The monomials associated to the geometry of the r-th trivalent vertex can be collected into three groups:

$$
\begin{aligned}
&x_r P_r(w), && P_r(w) = \sum_{i=0}^{k_r} p_i^r \, w^i \ , \\
&x_r z_r Q_r(w), && Q_r(w) = \sum_{i=0}^{m_r} q_i^r \, w^i \ , \\
&\rho_r = \mu_r \, x_{r-1} x_{r+1} \frac{z_r}{x_r} \ ,
\end{aligned}
$$

where x_{T+1} and $x_r, z_r, \ r = 1, \ldots, T$ are $2\,T+1$ coordinates for the base geometry and w is the fiber variable. The polynomials $P_r(w)$ and $Q_r(w)$ describe the r-th gauge system and its associated fundamental matter, respectively. Moreover p_i^r, q_i^r and μ_r are parameters related to the moduli of the combined gauge and matter system. The monomials combine to the hypersurface constraint

$$p(X^\star) = 1 + \sum_{r=1}^{T+1} x_r P_r(w) + \sum_{r=1}^{T} x_r z_r Q_r(w) + \sum_{r=1}^{T} \rho_r \ . \tag{6.4}$$

The coordinates z_r appear all linearly and integrating them out imposes a set of T constraint equations:

$$e_r : \quad x_r^2 Q_r(w) + \mu_r x_{r-1} x_{r+1} = 0 \ , r = 1, \ldots, T,$$

with solution

$$x_r = x^r \prod_{\alpha=1}^{r} (\tilde{Q}_\alpha)^{r-\alpha}(w), \qquad \tilde{Q}_r(w) = -\frac{1}{\mu_r} Q_r(w) \ .$$

Plugging them back into (6.4) we obtain the final mirror geometry, which describes again a Riemann surface:

$$p(X^\star) = 1 + \sum_{r=1}^{T+1} x^r P_r(w) \prod_{\alpha=1}^{r} (\tilde{Q}_\alpha)^{r-\alpha}(w) \ . \tag{6.5}$$

Exact Solution for the Coulomb Branch

The Riemann surface (6.5) has the geometrical interpretation of a single type IIA five-brane [8] as a result from a T-duality transformation on the ALE fiber space [11]. For the present gauge and matter content it was derived in [29] from strong-weak coupling duality between type IIA and M-theory. However the exact solution of the gauge theory needs more than just the geometry of the Riemann surface E, as is clear from the fact that the same E can describe very different kind of $N = 2$ theories depending on a choice of metric on E. These data can be obtained easily from the type IIB picture, which was the starting point of the T-duality to the five-brane in [8]. This is a straight-forward aspect of our toric construction: once the type IIA geometry is set up, the toric machinery proceeds in a stubborn and precise way until the very end, giving the exact solution of the theory in terms of period integrals on $X^\star$.

The moduli of the above gauge/matter system in terms of complex structure moduli of type IIB are given by

$$z_{r,i}^{(c)} = \frac{p_{i-1}^r p_{i+1}^r}{(p_i^r)^2}, \ i = 1, \ldots, k_r - 1$$

$$z_r^{(m),fund} = \frac{p_0^r q_1^r}{p_1^r q_0^r}, \ z_{r,i}^{(m),fund} = \frac{q_{i-1}^r q_{i+1}^r}{(q_i^r)^2}, \quad i = 1, \ldots, m_r - 1$$

$$z_r^{(m),bifund} = \frac{p_0^r p_1^{r+1}}{p_1^r p_0^{r+1}},$$

$$z_r^{(g)} = \frac{p_0^{r-1} p_0^{r+1} q_0^r}{(p_0^r)^2 \mu_r}$$

They describe the complex volumes of 2-spheres as follows: $z_{r,i}^{(c)}$ are the $k = \sum k_r - 1$ Coulomb parameters, $z_{r,i}^{(m),fund}$ and $z_r^{(m),fund}$ describe the $m = \sum m_r$ mass parameters for the fundamental matter and $z_r^{(m),bifund}$ describe the T mass parameters for the bi-fundamentals. Moreover $z_r^{(g)}$ are T coupling constants, parameterizing the $T - 1$ relative gauge couplings and the Planck mass.

For the derivation of the periods from the Picard-Fuchs equations, it is convenient to use a complete intersection description of the above geometry [47]. To do this, we have to give the so-called nef partition, which roughly speaking corresponds to finding subsets S_r of fields whose $U(1)$ charges q_i^a add up to zero within each set. For the present case this is trivial due to the linear appearance of the z_r in (6.4), and the nef partition is obtained by grouping the vertices and monomials by powers of the z_r. In this way we obtain a set S_0 containing all z_r independent monomials and T sets S_r containing the monomials linear in z_r for each r. The holomorphic $(3,0)$ form is then [48][43]

$$\Omega = \frac{1}{P_0} \prod_{r=1}^{T} \frac{1}{P_r} \prod_{i=1}^{T+4} \frac{dX_m}{X_m},$$

where the $T + 4$ variables X_m correspond to the base variables $x_r, r = 1, \ldots, T+1$, the fiber variable w, and two extra trivial variables v, u which we add quadratically to P_0 as in as in (3.3). Here P_0 is the Laurent polynomial related to the hypersurface constraint by setting $z_r = 0, \forall r$ in (6.4) and similarly the P_r correspond to the T equations e_r[12]. More precisely, the residue of Ω gives the holomorphic $(3,0)$ form on the threefold defined by $P_r = 0, \ r = 0, \ldots, T$.

---

[12] For the precise definition of Laurent polynomials and the coordinates X_m, see [48]. For an alternative canonical formulation using homogeneous coordinates, see [49].

It is straightforward to check that Ω fulfills the GKZ system of differential equations (4.18), which determine the instanton expansion for the period integrals of Ω.

Let us finally give the field theory limit and check that it agrees with the general expectations. Requiring to be near the singular point of each of the A factors of the fiber geometry, the limit is of the form

$$p_0^r \sim \epsilon^{-k_r}, \quad q_0^r \sim \epsilon^{-m_r}, \quad \mu_r \sim \epsilon^0 \ ,$$

implying finiteness of the field theory moduli

$$z_{r,i}^{(c)} \sim z_{r,i}^{(m),fund} \sim z_r^{(m),fund} \sim z_r^{(m),bifund} \sim \epsilon^0 \ .$$

On the other hand the scale variables $z_r^{(g)}$ behave as

$$z_r^{(g)} \sim \epsilon^{-b_r}, \qquad b_r = 2k_r - k_{r+1} - k_{r-1} - m_r \ ,$$

precisely as required by the beta-function coefficient b_r of the r-th gauge factor.

7. Affine E_n Geometries From the Trivalent Geometry

In the next section we will derive the mirror geometry of affine singularities using elliptic fibrations over the complex plane. Here we want to describe briefly, how we can get the affine E_n geometries from the trivalent geometry by extending what was done in the previous section.

Since we are interested in the base geometry to look like an affine E_n Dynkin diagram, we now allow all three A chains emerging from the central sphere have length > 1. Let us denote these chains as A_{p-1}, A_{q-1}, A_{r-1}. From (6.1) and the obvious intersections of the A chains, the mirror geometry contains the monomials

$$1, x, y, z, xyz; \ x^2, x^3, \ldots, x^p; \ y^2, y^3, \ldots, y^q; \ z^2, z^3, \ldots, z^r. \tag{7.1}$$

and describes the complex deformation of a singularity called $T_{p,q,r}$ [50]. From the discussion of the trivalent vertex in the previous section, it is clear what this corresponds to the mirror of type IIA geometry of trivalent base geometry.
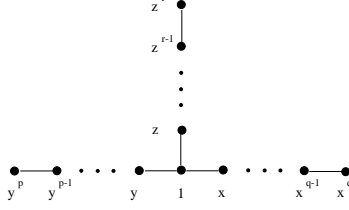
Fig. 11: The $T_{p,q,r}$ singularity.

The general singularity of this type has indefinite intersection form and will not lead to the perturbative prepotential of a well-defined field theory in four dimensions (though it would be of interest for questions of mirror symmetry in $N = 4$ theories in $d = 3$ as discussed later in the paper). As discussed in section 2 the limiting case that would be of interest for constructing superconformal theories in four dimensions are the ones (with semidefinite intersections)

$$T_{3,3,3} : y^3 + x^3 + z^3 + \mu xyz,$$
$$T_{2,4,4} : y^2 + x^4 + z^4 + \mu xyz, \qquad (7.2)$$
$$T_{2,3,6} : y^2 + x^3 + z^6 + \mu xyz ,$$

which give affine E_6, E_7 and E_8 geometry in the base. We can now proceed to the construction of the E_n type of superconformal theories as follows: we start with a type IIA singularity which is composed of three A_{n-1} chains of length $n_i = (p, q, r)$, which intersect the central sphere of a trivalent geometry, with (p, q, r) being one of the values in (7.2). For concreteness let us consider the E_6 case corresponding to $(p, q, r) = (3, 3, 3)$. The mirror geometry is described by the monomials (7.1), combined in the hypersurface constraint

$$p(X^\star) = v^3 + v^2(x + y + z) + v(x^2 + y^2 + z^2) + x^3 + y^3 + z^3 + xyz , \qquad (7.3)$$

where (x, y, z, v) are the homogeneous variables. Eq. (7.3) describes a del Pezzo surface B_6 with $c_1 \neq 0$. We can easily restore $c_1 = 0$ by using $v \cdot p(X^\star)$ as the hypersurface constraint, which describes a singular quartic $K3$ surface.

However this is not the end of the story because we wish to use this B_6 geometry as the base and fibering A_n chains above the blow up spheres of the original type IIA geometry. Let $k_{x,i}$ denote the rank of the A_n chain above the i-th base sphere in the x direction and similarly for y, z. The mirror geometry becomes

$$p(X^\star) = v^3 P_{k_{x,0}} + v^2(x P_{k_{x,1}} + y P_{k_{y,1}} + z P_{k_{z,1}}) +$$
$$v(x^2 P_{k_{x,2}} + y^2 P_{k_{y,2}} + z^2 P_{k_{z,2}}) + x^3 + y^3 + z^3 + xyz , \qquad (7.4)$$

41

where the coefficients $P_i = P_i(w, w')$ are polynomials of degree i in the homogeneous variables w, w' on the fiber. For

$$k_{x,i} = k_{y,i} = k_{z,i} = (3 - i)k, \ k \in \mathbf{Z},$$

the hypersurface determined by the polynomial $vww'p(X^\star) = 0$ describes a Calabi–Yau threefold $X^\star$. There are two independent $\mathbf{C}^\star$ actions $(y, x, z, v, w, w') \to (\lambda^{k-2}y, \lambda^{k-2}x, \lambda^{k-2}z, \lambda^{-2}\mu^{-k}v, \lambda\mu w, \lambda\mu w')$, which can be used to set v and w' to one.

The above geometry, and its E_7, E_8 variants based on the other two singularities in (7.2), describe the exact solutions to the superconformal $N = 2$ gauge theories defined in sect. 2. The toric geometry of the E_n base can be read off from fig. 12, where to each node we show the associated monomial of the mirror geometry and the Dynkin numbers which determine the relative multiplicities in the rank of the A_n fibers.
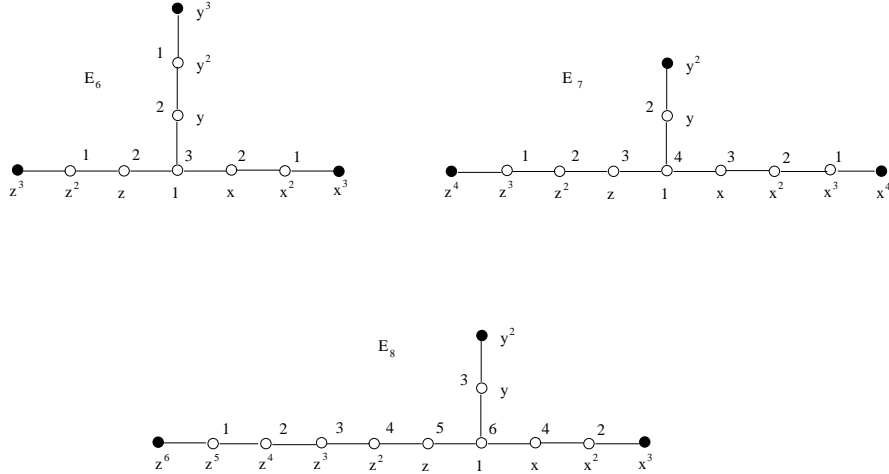

Fig. 12: Affine E_n base geometry of the 4d superconformal theories.

The fibration of the A_n fibers proceeds as in the E_6 case described above and leads to the

following defining polynomials for the mirror geometry:

$$E_6: \; SU(k)^3 \times SU(2k)^3 \times SU(3k)$$

$$p(X^\star) = y^3 + x^3 + z^3 + \mu xyz$$

$$+ \sum_{i=1}^{3} z^{3-i} v^i P^z_{i\cdot k}(w) + \sum_{i=1}^{2} x^{3-i} v^i P^x_{i\cdot k}(w) + \sum_{i=1}^{2} y^{3-i} v^i P^y_{i\cdot k}(w),$$

$$E_7: \; SU(k)^2 \times SU(2k)^3 \times SU(3k)^2 \times SU(4k)$$

$$p(X^\star) = y^2 + x^4 + z^4 + \mu xyz \tag{7.5}$$

$$+ \sum_{i=1}^{4} z^{4-i} v^i P^z_{i\cdot k}(w) + \sum_{i=1}^{3} x^{4-i} v^i P^x_{i\cdot k}(w) + \sum_{i=1}^{1} y^{2-i} v^{2i} P^y_{2i\cdot k}(w),$$

$$E_8: \; SU(k) \times SU(2k)^2 \times SU(3k)^2 \times SU(4k)^2 \times SU(5k) \times SU(6k)$$

$$p(X^\star) = y^2 + x^3 + z^6 + \mu xyz$$

$$+ \sum_{i=1}^{6} z^{6-i} v^i P^z_{i\cdot k}(w) + \sum_{i=1}^{2} x^{3-i} v^{2i} P^x_{2i\cdot k}(w) + \sum_{i=1}^{1} y^{2-i} v^{3i} P^y_{3i\cdot k}(w).$$

7.1. Exact Solution for the Coulomb Branch

We proceed with the exact solution of the superconformal theory corresponding to the affine E_8 quiver. The other two cases can be treated very similarly.

To fix notations, let us label the vertices of the type IIA geometry in fig.12 by the letters $a_{i,j}$ for the "z" leg and similarly by $b_{i,j}$ for the other two legs starting from the central sphere. The first subscript i denotes the Dynkin number and the second subscript the j-th vertex of the A_n chain of the fiber. The fiber polynomials in (7.5) are then

$$P^x_{2i\cdot k}(w) = \sum_{j=0}^{2i\cdot k} b_{i,j} w^j, \; i \in \{2,4\}, \quad P^y_{3k}(w) = \sum_{j=0}^{3k} b_{3,j} w^j,$$

$$P^z_{i\cdot k}(w) = \sum_{j=0}^{i\cdot k} a_{i,j} w^j, \; i = 1, \ldots, 6.$$

We abbreviate the parameters of the base by dropping the second subscript, $a_i \equiv a_{i,0}$, etc. In total we have $13 + c_2(E_8)k = 30k + 13$ vertices. To obtain a local threefold they should fulfill $30k + 8$ relations. These are the $30k - 9$ Coulomb parameters, 8 mass parameters,

the 8 relative coupling constants and the elliptic modulus:

$$z_i^{(g)} = \frac{a_{i+1}a_{i-1}}{a_i^2}, i = 1, ..., 5, \ a_0 \equiv 1, \quad z_6^{(g)} = \frac{a_5 b_3 b_4}{a_6^2 \mu},$$

$$y_2^{(g)} = \frac{b_4}{b_2^2}, \ y_3^{(g)} = \frac{a_6}{b_3^2}, \ y_4^{(g)} = \frac{a_6 b_2}{b_4^2},$$

$$z_i^{(c)} = \frac{a_{i,l-1}a_{i,l+1}}{a_{i,l}^2}, i = 1, ..., 6, \ l = 1, ..., ik - 1, \tag{7.6}$$

$$y_i^{(c)} = \frac{b_{i,l-1}b_{i,l+1}}{b_{i,l}^2}, i = 2, 3, 4, \ l = 1, ..., ik - 1,$$

$$z_i^{(m)} = \frac{a_{i,0}a_{i+1,1}}{a_{i,1}a_{i+1,0}}, i = 1, ..., 5, \ z_6^{(m)} = \frac{b_{4,0}b_{2,1}}{b_{4,1}b_{2,0}}, \ z_7^{(m)} = \frac{a_{6,0}b_{4,1}}{a_{6,1}b_{4,0}}, \ z_8^{(m)} = \frac{a_{6,0}b_{3,1}}{a_{6,1}b_{3,0}},$$

where we have used a basis adapted to the Dynkin diagram for the coupling constants $z_i^{(c)}$. The elliptic class is given by $\prod_{k=1}^{6}(z_k^{(g)})^k \prod_{k=2,3,4}(y_k^{(g)})^k = \mu^{-6}$.

Using the definition (4.16) of the complex structure moduli, it is easy to read off the charge vectors $l^{(a)}$ from (7.6). The exact instanton corrected prepotential is then given in terms of the period integrals, the solution to the Picard-Fuchs system (4.18). The field theory limit is as in eq.(5.6), that is the coefficient of the power of w^i of a $SU(k)$ factor scales as ϵ^{k-i}. It is easy to check that all moduli in (7.6) scale as ϵ^0 in this limit. Moreover the holomorphic form on X can be written in homogeneous variables as

$$\Omega = \frac{dy\,dx\,dz\,dv\,dw\,dw'}{vww'p(X^\star)},$$

where w' is the second homogeneous variable on the fiber. More precisely, taking into account the invariance under the two $\mathbf{C}^\star$ actions, Ω restricts to the holomorphic $(3, 0)$ form on the hypersurface $p(X^\star) = 0$. Very similar statements hold for the field theory limits and the holomorphic forms for the following theories based on affine base geometries and will not be repeated.

It is clear from the above that we can also obtain the curve for the case with arbitrary ranks of the gauge group corresponding to the affine E base, simply by changing the degree of the corresponding functions of w. We can also obtain the exact solution if we put additional fundamental matter for each gauge group, as we will next discuss.

7.2. D and E Dynkin Diagrams as the Base

As noted in section 2, asymptotic freedom also allows SU gauge theories based on ordinary D and E Dynkin diagrams. Note that the ordinary D and E Dynkin diagrams

also correspond to $T_{p,q,r}$ geometry. In particular D_n corresponds to the $T_{2,2,n-2}$ geometry and the ordinary E cases are the same as $T_{2,3,5}, T_{2,3,4}$ and $T_{2,3,3}$ for $E_{8,7,6}$ respectively. It is clear from the previous discussion how one writes the type IIB geometry for these cases with arbitrary rank gauge group on top of each node. Moreover, we can add fundamental matter to each gauge group just as in the case of linear chain. In fact treating each edge of the E_n or D_n Dynkin diagram emanating from the trivalent vertex as a linear chain, we have a situation already studied. In solving for the curve we will get just as in the linear chain case additional polynomials raised to some powers for each node, and their powers will increase as we go down the chain. Since various aspects of these have already been discussed in detail in previous sections we will not go into any further detail.

8. Elliptic Singularities and Affine ADE Quivers

We complete now the description of gauge groups with the base geometry given by affine ADE singularities using elliptic fibrations over the complex plane. This kind of singularities is well studied in the mathematics literature [28][18] and has been analyzed thoroughly in the context of heterotic/F-theory duality in [21].

The result from local mirror symmetry we obtain is that *the local mirror geometry of the Kähler resolution of affine ADE singularity describes the moduli space of flat ADE bundles on a torus E.* This was already anticipated from our discussion of section 3. Although we restrict again to the ADE case, the non-simply laced cases can be treated similarly [22].

We proceed in this section as follows. First we will study the missing cases, the two infinite series of 4d superconformal theories based on the affine A_n and D_n geometries, respectively. We will then relate our results for all the ADE geometries to the description of flat G bundles on elliptic curve E and use this connection to determine the S duality groups.

8.1. Affine A_n from Elliptic Fibrations over the Plane

The first of the two infinite series of superconformal 4d gauge theories consists of fibering A_k singularities over the affine A_N base. This base geometry can be constructed by blowing up the local singularity

$$y^2 + x^3 + x^2 + t^{N+1} \tag{8.1}$$

in a fibration of an elliptic curve E over the $\mathbf{C}$ plane. As before, we choose a sextic in $\mathbf{WP}_{1,2,3}$ as our model of E.

Let us describe the toric data for the type IIA base geometry. The polyhedron is defined by four vertices $\tilde{v}$, describing the generic sextic, and $N+1$ vertices v_i, $i = 1, ..., N+1$ introduced by the blow ups:

$$\tilde{v}_0 = (0,0,0), \tilde{v}_1 = (0,2,3), \tilde{v}_2 = (0,-1,0), \tilde{v}_3 = (0,0,-1),$$
$$v_1 = (1,2,3), v_{2i} = (1,2-i,3-i), v_{2i+1} = (1,2-i,2-i), \ i > 0 \tag{8.2}$$

Above each blow up sphere we fiber now an A_k singularity. To describe the toric polyhedron, we add a zero entry at the end of each vertex in (8.2) and join $(k+1) \cdot (N+1)$ further vertices:

$$v_{i,l} = v_i + (0,0,0,l), \ l = 1, ..., k+1, \tag{8.3}$$

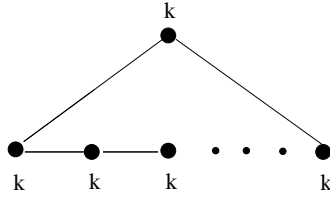This completes the construction of the type IIA geometry.

Fig. 13: $SU(k+1)^{N+1}$ gauge theory from affine A_N in the base.

As before, let us associate to each vertex $v_{i,l}$ the parameter $a_{i,l}$ that multiplies the corresponding monomial in the defining equation of the mirror geometry. Moreover we abbreviate $a_{i,0}$ by a_i. The relations $l^{(a)}$ needed to define the mirror geometry and its moduli space, written in terms of the moduli (4.16), are as follows: The volumes in the base are parametrized by the relations:

$$z_1^{(g)} = \frac{a_2 a_3 \tilde{a}_1}{a_1^2 \tilde{a}_0}, \quad z_2^{(g)} = \frac{a_1 a_4}{a_2^2}, \quad z_3^{(g)} = \frac{a_1 a_5 \tilde{a}_3}{a_3^2 \tilde{a}_0},$$

$$z_{2i+2}^{(g)} = \frac{a_{2i} a_{2i+4}}{a_{2i+2}^2}, \ i = 1, ..., \begin{cases} \frac{N}{2} - 2 & N \text{ even} \\ \frac{N+1}{2} - 2 & N \text{ odd} \end{cases},$$

$$z_{2i+1}^{(g)} = \frac{a_{2i-1} a_{2i+3}}{a_{2i+1}^2}, \ i = 2, ..., \begin{cases} \frac{N}{2} - 1 & N \text{ even} \\ \frac{N+1}{2} - 2 & N \text{ odd} \end{cases}, \tag{8.4}$$

$$z_N^{(g)} = \frac{a_{N-2} a_{N+1}}{a_N^2} \times \begin{cases} \frac{\tilde{a}_2}{\tilde{a}_0} & N \text{ even} \\ \frac{\tilde{a}_3}{\tilde{a}_0} & N \text{ odd} \end{cases}, \quad z_{N+1}^{(g)} = \frac{a_N a_{N-1}}{a_{N+1}^2} \times \begin{cases} \frac{\tilde{a}_2 \tilde{a}_3^2}{\tilde{a}_0^3} & N \text{ even} \\ \frac{\tilde{a}_3 \tilde{a}_2^2}{\tilde{a}_0^3} & N \text{ odd} \end{cases}$$

Moreover, the gauge system of the fiber is described by $k \cdot (N+1)$ Coulomb moduli and $N+1$ mass parameters, of which N are independent:

$$z_{i,l}^{(c)} = \frac{a_{i,l-1}a_{i,l+1}}{a_{i,l}^2}, \qquad z_{i,l}^{(m)} = \frac{a_{i,0}a_{i+2,1}}{a_{i,1}a_{i+2,0}}. \tag{8.5}$$

In total we have $n_\nu = 4 + (N+1) \cdot (k+2)$ vertices fulfilling $n_R = 1 + N + (N+1) \cdot k + N$ relations. Taking into account the hypersurface constraint this gives a mirror geometry of dimension $n_\nu - n_R - 2 = 3$. Combining the monomials that solve (4.13) with the $l^{(a)}$ as defined by the above relations and (4.16), the hypersurface constraint reads:

$$A_N \ : \ SU(k+1)^{N+1}$$

$$p(X^\star) = v(y^2 + x^3 + z^6 + \mu yxz)$$
$$+ z^{N+1} P_{k+1}^{(N+1)}(w) + z^{N-1} x P_{k+1}^{(N-1)}(w) + z^{N-2} y P_{k+1}^{(N-2)}(w) + \dots \tag{8.6}$$
$$+ \begin{cases} yx^{\frac{N-2}{2}} P_{k+1}^{(0)}(w) & N \text{ even} \\ x^{(N+1)/2} P_{k+1}^{(0)}(w) & N \text{ odd} \end{cases}$$

The coefficients in the polynomials $P_{k+1}^{(K)}(w)$ are related to those in eqs.(8.4),(8.5) by $P_{k+1}^{(K)}(w) = \sum_{l=0}^{k+1} a_{N+1-K,l} \, w^l$.

To solve $p(X^\star)$, note that v appears only linearly and can be integrated out. This results in the constraint:

$$E : \ y^2 + x^3 + z^6 + \mu yxz = 0 \ .$$

Thus (y, x, z) become coordinates on an elliptic curve E. We are left with the second term in $p(X^\star)$, which, after reordering in powers of w, reads

$$0 = \sum_{i=1}^{k+1} \tilde{f}_i(y, x, z) w^i = \tilde{f}_{k+1}(y, x, z) \left(w^{k+1} + \sum_{i=1}^{k} f_i(y, x, z) \, w^i \right) , \tag{8.7}$$

where for N odd

$$\tilde{f}_i(y, x, z) = a_{1,i} z^{N+1} + a_{2,i} z^{N-1} x + \dots + a_{N+1,i} x^{(N+1)/2}$$
$$f_i(y, x, z) = \tilde{f}_i(y, x, z) \, \tilde{f}_{k+1}^{-1}(y, x, z), \tag{8.8}$$

and similarly for N even.

The functions f_i are rational functions on the torus E, with poles at the zeros of f_{k+1}. This is in agreement with the results in [29], where the zeros of f_{k+1} are interpreted as the positions of five-branes on the torus E.

8.2. Affine D_N from Elliptic Fibrations over the Plane

The second infinite series arises from a base geometry of 2-spheres intersecting as determined by the affine Dynkin diagram of D_n. The local singularity which is blown up on the type IIA side is given by

$$y^2 = x^2(x + ct) + t^{N-1}, \tag{8.9}$$

A description of the type IIA geometry based on the standard representation of the torus as a sextic in $\mathbf{WP}_{1,2,3}$ starts from a toric polyhedron Δ spanned by the vertices for the elliptic curve

$$E : \rho_1 = (0, 0, -1), \;\; \rho_2 = (0, -1, 0), \;\; \rho_3 = (0, 2, 3), \;\; \nu_{a_0} = (0, 0, 0), \tag{8.10}$$

together with $N + 1$ vertices describing the blow up spheres of the singularity (8.9):

$$\nu_1^b = (1, 1, 1), \; \nu_2^b = (1, 2, 3), \nu_i^a = (2, 3 - i, 4 - i), \;\; i = 1, ..., N - 3$$

$$N \;\; \text{even} : \begin{cases} \nu_1^c = (1, 2 - n, 2 - n), \\ \nu_2^c = (1, 1 - n, 2 - n), \end{cases}$$

$$N \;\; \text{odd} : \begin{cases} \nu_1^c = (1, 1 - n, 1 - n), \\ \nu_2^c = (1, 1 - n, 2 - n), \end{cases}$$

where $n = \frac{1}{2}(N - 2)$ for N even and $n = \frac{1}{2}(N - 3)$ for N odd. We use again the definition of the moduli (4.16) to describe the charge vectors $l^{(a)}$, needed to determine the mirror geometry:

$$z_i^{(g)} = \frac{a_i a_{i+2}}{a_{i+1}^2}, \;\; i = 1, ..., N - 5,$$

$$y_1^{(g)} = \frac{a_2 b_1 b_2}{a_1^2 a_0}, \quad y_2^{(g)} = \frac{a_{N-4} c_1 c_2}{a_{N-3}^2 a_0},$$

$$x_1^{(g)} = \frac{a_1 \rho_1}{b_1^2}, \quad x_2^{(g)} = \frac{a_1 \rho_3}{b_2^2}, \tag{8.11}$$

$$\begin{cases} x_3^{(g)} = \frac{a_{N-3} \rho_1}{c_1^2}, \quad x_4^{(g)} = \frac{a_{N-3} \rho_2^2 \rho_1}{c_2^2 a_0^2} & N \text{ even}, \\ x_3^{(g)} = \frac{a_{N-3} \rho_2}{c_2^2}, \quad x_4^{(g)} = \frac{a_{N-3} \rho_2 \rho_1^2}{c_1^2 a_0^2} & N \text{ odd}. \end{cases}$$

The elliptic class is given by $(\prod z_i^{(g)})^2 (\prod y_i^{(g)})^2 (\prod x_i^{(g)})$ with the powers reflecting the Dynkin numbers of the affine root. The mirror geometry, obtained by solving (4.13), results in the hypersurface constraint

$$D_N : \quad p(X^\star) = p_0 + v p_1 + v^2 p_2, \tag{8.12}$$

48

with

$$p_0 = (y^2 + x^3 + z^6 + a_0 xyz),$$

$$p_1 = (b_1 z^N y + b_2 z^{N+3} + c_1 z^{4-\epsilon} yx^{(N+\epsilon)/2-2} + c_2 z^{3+\epsilon} x^{(N-\epsilon)/2}),$$

$$p_2 = (a_1 z^{2N} + a_2 z^{2N-2} x + a_3 z^{2N-4} x^2 + ... + a_{N-3} z^8 x^{N-4}).$$

where $\epsilon = 0$ (1) for N even (odd). The hypersurface constraints $p(X^\star)$ are invariant under the $\mathbf{C}^\star$ action $(y, x, z, v) \rightarrow (\lambda^3 y, \lambda^2 x, \lambda z, \lambda^{3-N} v)$ and describe two-complex dimensional Calabi–Yau hypersurfaces, after appropriate multiplication with powers of v.

To construct the 4d superconformal field theory we use this geometry as the base geometry and then fiber, on the type IIA side, $A_{n'}$ chains over each 2-sphere in the base, as dictated by the Dynkin numbers of affine D_n, times an overall integer k.
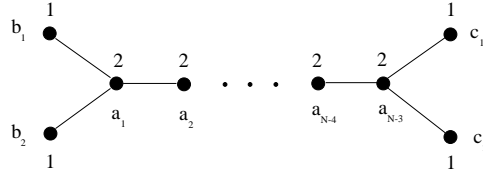

Fig. 14: Toric diagram for D_N : Dynkin numbers and conventions.

As before, we describe these $A_{n'}$ chains by adding new vertices to the polyhedron Δ. They are

$$\begin{aligned}
\nu^a_{i,j} &= (*, *, *, j), & i &= 1, ..., N-3, & j &= 0, ..., 2k+1, & \nu^a_{i,0} &\equiv \nu^a_i. \\
\nu^b_{i,j} &= (*, *, *, j), & i &= 1, 2, & j &= 0, ..., k+1, & \nu^b_{i,0} &\equiv \nu^b_i. \\
\nu^c_{i,j} &= (*, *, *, j), & i &= 1, 2, & j &= 0, ..., k+1, & \nu^c_{i,0} &\equiv \nu^c_i.
\end{aligned}$$

and we denote again the moduli multiplying the monomials of the mirror geometry corresponding to these vertices by $a_{i,j}$, $b_{i,j}$ and $c_{i,j}$, respectively.

To determine the dimension of x, note that we had $N+5$ vertices from the base and we add now $(N-1)(2k+1)+2$ further ones. Moreover we had $N+1$ coupling parameters measuring volumes in the base geometry. A 3-fold geometry requires $2(N-1)k+N$ further relations. These are the $2(N-1)k$ Coulomb fields $z^{(c)}$ and N mass parameters $z^{(m)}$:

$$\begin{aligned}
z^{(c)}_{a,i,j} &= \frac{a_{i,j} a_{i,j+2}}{a_{i,j+1}^2}, & i &= 1, ..., N-3, & j &= 0, ..., 2k-1 \\
z^{(c)}_{b,i,j} &= \frac{b_{i,j} b_{i,j+2}}{b_{i,j+1}^2}, & i &= 1, 2, & j &= 0, ..., k-1 \\
z^{(c)}_{c,i,j} &= \frac{c_{i,j} c_{i,j+2}}{c_{i,j+1}^2}, & i &= 1, 2, & j &= 0, ..., k-1 \\
z^{(m)}_{a,i} &= \frac{a_{i,0} a_{i+1,1}}{a_{i,1} a_{i+1,0}} & i &= 1, ..., N-4, \\
z^{(m)}_{b,i} &= \frac{a_{1,0} b_{i,1}}{a_{1,1} b_{i,0}}, & i &= 1, 2, \\
z^{(m)}_{c,i} &= \frac{a_{N-3,0} c_{i,1}}{a_{N-3,1} c_{i,0}}, & i &= 1, 2,
\end{aligned} \tag{8.13}$$

49

The mirror geometry describing the geometry of the 4d $N = 2$ field theory is then given in terms of the hypersurface (8.12) with

$$
\begin{aligned}
D_N \ : \ & SU(k+1)^4 \times SU(2k+1)^{N-3} \\
p_0 = & (y^2 + x^3 + z^6 + a_0 xyz), \\
p_1 = & (b_1 z^N y P^{b_1}_{k+1}(w) + b_2 z^{N+3} P^{b_2}_{k+1}(w) + \\
& c_1 z^{4-\epsilon} yx^{(N+\epsilon)/2-2} P^{c_1}_{k+1}(w) + c_2 z^{3+\epsilon} x^{(N-\epsilon)/2} P^{c_2}_{k+1}(w)), \\
p_2 = & (a_1 z^{2N} P^{a_1}_{2k+1}(w) + a_2 z^{2N-2} x P^{a_1}_{2k+1}(w) + ... + a_{N-3} z^8 x^{N-4} P^{a_{N-3}}_{2k+1}(w)),
\end{aligned}
\tag{8.14}
$$

where again $\epsilon = 0$ (1) for N even (odd) and the polynomials in the fiber variable are defined as

$$
P^{b_1}_{k+1}(w) = \sum_{j=0}^{k+1} b_{1,j} w^j,
$$

and similarly for the other terms.

Eqs. (8.11) and (8.13) define the charge vectors $l^{(a)}$. The exact solution is then given in terms of the Picard-Fuchs system (4.18).

8.3. Moduli space of G bundles on elliptic curves E

We will describe now the relation of the mirror geometry of elliptic ADE singularities to the geometric representation of moduli spaces of flat ADE bundles over an elliptic curve. Physically, these moduli space of G bundles over elliptic curve E is interesting in the light of the duality between heterotic string on elliptically fibered manifolds and F-theory. This has been studied in [19] and [51].

Let us collect the equations defining the two complex dimensional mirror geometry of the base manifold, adding the E_n cases, which can be obtained in a similar way from

fibering the elliptic singularities over the plane:

$$\mathbf{A_{N-1}} : (y^2 + x^3 + z^6 + \mu yxz) + v(a_1 z^N + a_2 z^{N-2} x + a_3 z^{N-3} y + ... + \left\{ \begin{matrix} a_N x^{N/2} \\ a_N yx^{\frac{N-3}{2}} \end{matrix} \right\}),$$

$$\mathbf{D_N} : (y^2 + x^3 + z^6 + \mu xyz)$$
$$+ v(b_1 z^N y + b_2 z^{N+3} + c_1 z^{4-\epsilon} yx^{(N+\epsilon)/2-2} + c_2 z^{3+\epsilon} x^{(N-\epsilon)/2})$$
$$+ v^2(a_1 z^{2N} + a_2 z^{2N-2} x + a_3 z^{2N-4} x^2 + ... + a_{N-3} z^8 x^{N-4}),$$

$$\mathbf{E_6} : (y^2 z + x^3 + z^3 z'^6 + \mu xyz'z) + (a_2 v^2 xz'^4 + a_1 vx^2 z'^2)$$
$$+ (b_2 v^2 yz'^3 + b_1 vy^2) + (c_2 v^2 zz'^6 + c_1 vz^2 z'^6) + d_3 v^3 z'^6,$$

$$\mathbf{E_7} : (y^2 + x^3 z + z^4 z'^6 + \mu xyz'z) + (a_1 vx^3 + a_2 v^2 x^2 z'^2 + a_3 v^3 z'^4 x)$$
$$+ (b_3 v^3 z'^6 z + b_2 v^2 z'^6 z^2 + b_1 vz'^6 z^3) + c_2 v^2 z'^3 y + d_4 z'^6 v^4,$$

$$\mathbf{E_8} : (y^2 + x^3 + z^6 + \mu xyz)$$
$$+ (a_6 v^6 + a_5 v^5 z + a_4 v^4 z^2 + a_3 v^3 z^3 + a_2 v^2 z^4 + a_1 vz^5)$$
$$+ (b_3 v^3 y + b_2 v^2 x^2 + b_4 v^4 x).$$

$$(8.15)$$

We assert that the complex deformations of the above two dimensional surfaces give a geometrical representation of the moduli space of flat G bundles over an elliptic curve E, where G is one of the above ADE groups. To this end note that the divisor $v = 0$ projects onto the degree six elliptic curve $E : \mathbf{WP}^2_{1,2,3}$ with modulus μ. Moreover the scaling $v \to \lambda v$ induces a projective action on the moduli parameterizing the complex structure, such that they become coordinates on the weighted projective spaces:

$$A_{N-1} : (a_1, ..., a_N) \in \mathbf{P}^{N-1},$$

$$D_N : (b_1, b_2, c_1, c_2, a_1, ..., a_{N-3}) \in \mathbf{WP}^N_{1,1,1,1,2,...,2},$$

$$E_6 : (a_1, b_1, c_1, a_2, b_2, c_2, d_3) \in \mathbf{WP}^6_{1,1,1,2,2,2,3},$$

$$E_7 : (a_1, b_1, a_2, b_2, c_2, a_3, b_3, d_4) \in \mathbf{WP}^7_{1,1,2,2,2,3,3,4},$$

$$E_8 : (a_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, a_6) \in \mathbf{WP}^8_{1,2,2,3,3,4,4,5,6}.$$

These are precisely the moduli spaces predicted by Looijenga's analysis of the moduli space of flat G bundles on E [52].

Let us describe the geometry of these spaces in more detail. In the A_N case, v appears only linearly, and integrating it out results in setting the v^0 and v^1 pieces to zero separately. The result is a zero dimensional geometry of $N + 1$ points on the elliptic curve E. This is the spectral cover description presented in [19][51]. For the E_n cases, setting the extra

variable z' to one in virtue of the present $\mathbf{C}^\star$ symmetries, the equations (8.15) represent del Pezzo surfaces[13] $E_8 : \mathbf{WP}^3_{1,1,2,3}[6]$, $E_7 : \mathbf{WP}^3_{1,1,1,2}[4]$, $E_6 : \mathbf{WP}^3_{1,1,1,1}[3]$, respectively. Again this agrees with the representation of E_n bundles in terms of complex deformations of del Pezzo surfaces in ref. [19].

This identifies the apparently disconnected descriptions of the geometrical objects appearing in the analysis of [19] as the mirror of the *physical type IIA compactification geometry*. Note that the reasoning given in section 3 explains the connection we have found between the mirror of this type IIA geometry and the moduli of flat ADE bundles on an elliptic curve. Moreover here we have obtained also the geometrical description for the D_N case for which no representation in terms of complex deformations was known. As mentioned, the non-simply laced cases can be obtained in the same way [22].

8.4. S-duality Groups

As discussed in the previous sections, when we consider the chain of SU groups arranged according to the affine ADE Dynkin diagrams whose ranks are proportional to the Dynkin indices and where for each link we associate bi-fundamental matter, we obtain an $N = 2$ theory in four dimensions with vanishing β-function. Similarly if we consider the configuration of SU groups according to ordinary ADE diagram with extra matter fields as described previously, we obtain again superconformal theories. Thus the coupling constants of these gauge groups do not run and it is natural to ask what is the space of inequivalent coupling constants. This moduli space is the naive classical moduli space of coupling constants modulo the S-duality group.

From our construction of the type IIA geometry it is clear that the space of couplings is the same as the moduli controlling the (affine) ADE geometry of the base. Moreover the moduli space of the affine base geometry describes the blow up of elliptic ADE singularities and is equivalent to the moduli $\mathcal{M}_G$ of flat $G \subset ADE$ bundles on a 2-torus E, as we discussed in section 3 and the present section. *The S-duality group is then simply the fundamental group of $\mathcal{M}_G$.* In the case of ordinary ADE diagrams, the S-duality group can be obtained by degenerating the moduli of the elliptic curve in the corresponding affine case and the S-duality group is the subgroup of the affine one, which corresponds to moduli of flat ADE connection on the degenerate elliptic curve.

---

[13] The number in the last bracket denotes the degree of the polynomial.

The case corresponding to the A diagrams was already considered in [29][14], where the moduli space was shown to be the moduli of n points on an elliptic curve or its degeneration depending on whether we are dealing with the affine or ordinary A case. This is in agreement with our result when one notices that flat bundles of A-type on a torus are equivalent to the choice of n points on the dual torus or its degeneration.

It is quite suggestive that a gauge group associated with the base arises in describing this moduli space. This begs for a more direct physical interpretation, and is related to the strong coupling phenomena associated to shrinking the base, as we will discuss now.

9. Strong Coupling Fixed Points and New Dualities

We have seen that in our construction the base and fiber play a similar role. For example consider the curve we have for the linear chain of $SU(n)^m$ gauge theories along a linear chain with n additional fundamentals at each end of the chain. The corresponding threefold for this case is given by

$$F(z, w) = \sum_{i=0}^{m+1} \sum_{j=0}^{n} a_{i,j} \, z^i \, w^j = uv, \tag{9.1}$$

where z corresponds to the base degree of freedom and w to the fiber, as before. Clearly this geometry is invariant under the exchange of $n \leftrightarrow m + 1$ and $z \leftrightarrow w$, which would correspond to the geometry associated with $SU(m + 1)^{n-1}$ along the linear chain with $m + 1$ extra fundamentals at each end. In particular in either case the relevant fivebrane lives on the same genus $m \, (n - 1)$ Riemann surface. This suggests a "duality" of the form

$$SU(n)^m \leftrightarrow SU(m + 1)^{n-1} \tag{9.2}$$

with the matter described above, where the Coulomb parameters, the couplings and the masses get exchanged in a non-trivial fashion.
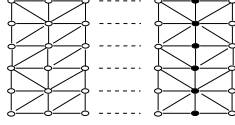
---

[14] See also [53].

Fig. 15: Toric diagram for the field theories (9.2) connected by the strong coupling fixed point.

Before we analyze the relation in (9.2) in more detail, let us use it to get some more insight about the S duality group of each one of these theories. The S-duality group describes the equivalences of the moduli space in the high energy behavior, or equivalently, the behavior for small values of the moduli. Neglecting the terms of lower degree in w in (9.1) (for the $SU(n)^m$ theory), $F(z,w)/w^n$ is of the form

$$\sum_{i=0}^{m+1} a_{i,n} z^i.$$

The moduli space is described by the m Coulomb moduli[15] $z^{(c)}$ of the $SU(m+1)$ theory corresponding to the base geometry. In particular, the effective duality group acting on the Coulomb parameters of $SU(m+1)$ generates the S-duality transformations of the $SU(n)$ theory.

As an explicit example consider the $m=1$ case. The coupling space describes the roots of an quadratic equation which, written in terms of the $SU(2)$ modulus $z^{(c)} = a_{2,n} a_{0,n} a_{1,n}^{-2}$ reads $F/w^n = z'^2 + (4z^{(c)} - 1)$. The singular points are at the values $z^{(c)} = 0, \frac{1}{4}, \infty$. The singularity $z^{(c)} = 0$, the large base limit, corresponds to weak coupling limit whereas $z^{(c)} = \infty$ is a $\mathbf{Z}_2$ orbifold point, as discussed in [29] in the M-theory picture. The five brane description breaks down at the point $z = \frac{1}{4}$ corresponding to a collapse of two five branes. In the type IIA geometry, this singularity describes simply the zero volume point of the blow up sphere of the A_1 base geometry.

In the general $SU(m+1)$ case there are m singular points of this type in the generic hyperplane of the moduli space. The S-duality group is generated by loops around these singular points together with the weak coupling monodromies generating the translations in the special coordinate $t \sim \frac{1}{2\pi i} \ln z^{(c)}$. Analogous statements apply to the coupling space of the D and E cases, as is clear from the above discussion and also from the large w limit of the polynomials given in the previous sections.

---

[15] More precisely there are finite shifts proportional to the mass parameters which are straightforwardly to determine from eqs. (5.5).

Let us determine now more carefully the precise relation between the theories appearing in (9.2). The local geometry of the type IIB side and its periods describe part of the exact moduli space of the $N = 2$ string theory, which is of special Kähler type. To reduce to the field theory we have on the one hand to decouple gravity. Moreover also the bare parameters of the field theory, the coupling constants and the mass parameters, are scalar fields which sit in full vector multiplets. Clearly we have to freeze out the fluctuations of these fields, if we are interested in the pure field theory answer described by an action including renormalizable couplings only.

In the asymptotic free case, the $M_{pl} \to \infty$ limit requires to adjust the coupling constants to zero at the Planck scale [5], in order to keep the field theory scale Λ at a fixed, finite value. Moreover, the vector multiplets corresponding to the bare parameters freeze out due to the behavior of their kinetic terms, whereas the vector fields of the gauge theory remain dynamical, if the moduli are tuned to to the neighborhood of the enhanced gauge symmetry point in moduli space.

On the contrary, in the case with vanishing β function, the couplings do not run above the natural scale of the field theory set by the vev's and the volume of the base remains unfixed in the $M_{pl} \to \infty$ limit. We have still to adjust the field theory moduli to be close to the enhanced symmetry point. However in this case it is possible to treat the fiber geometry in the same way as the base and we obtain a *new* low energy theory containing the coupling constants as additional dynamical fields. In particular note that although the curve (9.1) is as expected in gauge theory, the differential Ω is now symmetric in the base variable z and the fiber variable w and agrees with the field theory answer only after a shift the variable $w \to \mathcal{O}(\epsilon^{-1}) + w$:

$$\Omega = \ln(w)d\ln(z) = -d\ln(w) \ \ln(z)$$

$$= -\ln \epsilon \frac{dz}{z} + \epsilon w \frac{dz}{z} + \sum (-)^{k-1} \epsilon^k \frac{w^k}{k} \frac{dz}{z}$$

In the asymptotic free case, the first term corresponding to the period associated with the base volume is related to the weak gravity limit by $e^{-S} = (\Lambda^2 \alpha')^l = \epsilon^l$, where S is the dilaton determining the string coupling, $\alpha' \sim M_{string}^{-2}$ and l depends on the theory we consider. The 1-form proportional to ϵ in the above expression is what the gauge system sees. In the conformal case, the exponent l is zero and the scale ϵ is a new scale governing the dynamics of the additional fields from the base. It is clear that we can now switch the roles of the base and the fiber in the above limit. In fact we can continuously interpolate between the theories (9.2).

Of course these new theories symmetric in the base and fiber geometry are very interesting to study further. In order to gain insight into these cases, let us consider the closely related, but simpler case considered in [54][55][12]: If we consider $\mathbf{P}^1 \times \mathbf{P}^1$ in a Calabi-Yau, and view one of the $\mathbf{P}^1$'s as the fiber, we obtain a theory with pure $SU(2)$ gauge symmetry where the volume of the base $\mathbf{P}^1$ is related to the gauge coupling of the $SU(2)$, $V \sim 1/g^2$. However if we exchange the role of base and fiber we get the base $\mathbf{P}^1$ giving rise to $SU(2)$ and the volume of the fiber $\mathbf{P}^1$ is related to its coupling. So in a sense we have an $SU(2) \times SU(2)$ here, where the Coulomb space of either $SU(2)$ is identified with the coupling constant of the other $SU(2)$. This suggests couplings roughly of the form

$$\mathrm{tr} F_1^2 f(\phi_2) + (1 \leftrightarrow 2),$$

where ϕ_i denote the scalar adjoints of the two $SU(2)$'s and where f vanishes as $\phi_2 \to 0$. This theory is clearly not renormalizable, but the critical point corresponding to a superconformal theory where roughly speaking we are at the origin of the Coulomb branch for both $SU(2)$'s is known to exist [54][55].

The situation we are considering above is roughly of the same type, where now the Coulomb parameters of the base A_m play the role of the couplings of the various SU gauge groups in the fiber. But now the classical Weyl group of A_m exchanges the various SU gauge groups with each other thus the coupling of these two systems makes sense only if we consider the part of the S-duality group which exchanges the SU's. So in some sense it is like gauging the quantum symmetries of the theory. Clearly this is a very interesting area to study further.

Note also that the affine ADE bases that we have considered, if we allow shrinkings of the base, would correspond to new critical theories, and in fact correspond to compactification of M-theory on the same manifold times a circle, or F-theory on the elliptic version of the same manifold times a 2-torus [56][57][58][59][60]. Thus our results give answer to the Coulomb branch of such 6 dimensional critical theories compactified on T^2.

10. Applications to $d = 3$, $N = 4$ QFT's

Some of the results we have obtained have been used in [13] to derive dual pair of field theories with $N = 4$ in $d = 3$. This is done by considering a further compactification of our mirror IIB model and considering the Higgs branch and using T-duality on the extra circle and converting it to a IIA model Coulomb branch, compactified on a circle,

and reading off the gauge theory and matter content. Even though in this paper we have mostly concentrated on theories which are asymptotically free in $d = 4$ (ignoring the $U(1)$ factors), we already mentioned how one would construct the corresponding geometry even for non-asymptotically free theories. For applications in three dimensional QFT's the cases which are not asymptotically free in $d = 4$ (but which automatically are asymptotically free in $d = 3$) are more relevant as those are the cases that in cases which is completely Higgsable and which can have a dual gauge theory with Higgs and Coulomb branches exchanged [14][61][62].

It is clear from the approach in [13] that a dual system for any $N = 4$, $d = 3$ gauge system will exist if it comes from geometric engineering, however the dual may not be a gauge system. This in particular was shown to be the case for the dual of $\prod U(s_i)$ groups associated to an affine $E_{8,7,6}$ Dynkin diagram, where s_i are the Dynkin indices. The dual in this case was found to be the toroidal compactification of one $E_{8,7,6}$ small instanton in accordance with the conjecture [14]. The method used there was to start from the Coulomb branch of the compactified exceptional strings in the type IIB setup which was known [63] and then considering the Higgs branch of it and reading it as type IIA theory. However we can study the same problem in a completely different way, using the results of the present paper. Namely, we can start from the type IIB realization of the Coulomb branch of the SU groups associated with the affine E quivers and see if that can be viewed as the Higgs branch of the small E_n instantons, which is known [64][65] for E_8 theory, but the generalization to the other cases goes through without any complications.

Note that here it is crucial that we are actually dealing with $\prod U(s_i)$ rather than $\prod SU(s_i)$. As noted before the extra $U(1)$'s do not affect the Coulomb branch we studied in four dimensions and in fact the vev of the scalars in the $U(1)$ correspond to the mass parameters of the bi-fundamental matter. However they are crucial in the three dimensional story as the $U(1)$'s are asymptotically free in $d = 3$. Let us recall the local type IIB geometry which gives the Coulomb branch of this theory for the more general case of $\prod U(ks_i)$ (7.5):

$$E_8 : \ p(X^\star) = y^2 + x^3 + z^6 + \mu xyz + \sum_{i=1}^{6} z^{6-i} P_{i\cdot k}^z(w) +$$

$$\sum_{i=1}^{2} x^{3-i} P_{2i\cdot k}^x(w) + \sum_{i=1}^{1} y^{2-i} P_{3i\cdot k}^y(w)$$

Let us restrict first to $k = 1$. Furthermore, we can shift x, y so that the equation will involve only the monomials $y^2, x^3, x, 1$, with coefficients a function of z, w. Once we do this we find that the local geometry can be written as

$$
\begin{aligned}
y^2 =& x^3 + z^6 + axz^4 + x(z^3 f_1(w) + z^2 f_2(w) + z f_3(w) + f_4(w)) \\
& + (z^5 g_1(w) + z^4 g_2(w) + \ldots + g_6(w))
\end{aligned}
$$

where $f_i(w), g_i(w)$ are polynomials of degree i in w (which can be written in terms of the original polynomials). This is exactly the description of the Higgs branch of one small E_8 instanton in F-theory, which when compactified on T^2 becomes the description of type IIA on the same geometry, thus showing that the dual of one E_8 instanton compactified to three dimensions is the $\prod U(s_i)$ along the affine E_8 quiver diagram. The generalization to $U(ks_i)$ is straightforward: All that happens is that one shifts the degrees of polynomials $f_i, g_i \rightarrow f_{k \cdot i}, g_{k \cdot i}$ in the above equation, and that is what one expects for k instantons of E_8 [66][65]. This is also in accord with the conjecture [14].

Note that our approach will also allow us to find new dual systems. For example suppose we are interested in constructing the dual to $U(s_i)$ along the affine E_8 Dynkin diagram and in addition some extra fundamental matter for each group. Then the methods of the previous section can be used to give the geometry for the 3d dual type IIA Higgs branch, just as in the case considered. Then, however, it will not be related to any known small instantons theory, but the dual system will always exist, as was noted in [13].

References

[1] S. Kachru and C. Vafa, Nucl. Phys. $\underline{B450}$ (1995) 69

[2] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. $\underline{B357}$ (1995) 313

[3] C. Vafa and E. Witten, *Dual string pairs with N=1 and N=2 supersymmetry in four-dimensions*, hep-th/9507050

[4] J. Louis and P. Aspinwall, Phys. Lett. $\underline{B369}$ (1996) 233

[5] S. Kachru et. al., Nucl. Phys. $\underline{B459}$ (1996) 537

[6] N. Seiberg and E. Witten, Nucl. Phys. $\underline{B426}$ (1994) 19, erratum: ibid $\underline{B430}$ (1994) 396; Nucl. Phys. $\underline{B431}$ (1994) 484.

[7] A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, Phys. Lett. $\underline{B344}$ (1995);
P. Argyres and A. Faraggi, Phys. Rev. Lett. $\underline{73}$ (1995) 3931 ;
A. Hanany and Y. Oz, Nucl. Phys. $\underline{B452}$ (1995) 283;
P. Argyres, M. Plesser, and A. Shapere, Phys. Rev. Lett. $\underline{75}$ (1995) 1699;
U. Danielsson and B. Sundborg, Phys. Lett. $\underline{B358}$ (1995) 273;
A. Brandhuber and K. Landsteiner, Phys. Lett. $\underline{B358}$ (1995) 73;
A. Hanany, Nucl. Phys. $\underline{B466}$ (1996) 85;
P. C. Argyres and A. D. Shapere, Nucl. Phys. $\underline{B461}$ (1996) 437;
W. Lerche and N. Warner, *Exceptional SW Geometry from ALE Fibrations*, hep-th/9608183;
K. Landsteiner, J. M. Pierre, S. B. Giddings, Phys. Rev. $\underline{D55}$ (1997) 2367;
E. Martinec and N. Warner, Nucl. Phys. $\underline{B459}$ (1996) 97;
E. D'Hoker, I.M. Krichever and D.H. Phong, Nucl. Phys. $\underline{B489}$ (1997) 211

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. $\underline{B477}$ (1996) 746

[9] W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origin*, hep-th/9611190;

A. Klemm, *On the geometry behind N=2 supersymmetric effective actions in four-dimensions*, hep-th/9705131

[10] A. Brandhuber and S. Stieberger, Nucl. Phys. $\underline{B488}$ (1997) 177;
J. Schulze and N. P. Warner, *BPS geodesics in N=2 supersymmetric Yang-Mills theory*, hep-th/9702012;
J. M. Rabin, *Geodesics and BPS states in N=2 supersymmetric QCD*, hep-th/9703145

[11] H. Ooguri and C. Vafa, Nucl. Phys. $\underline{B463}$ (1996) 55

[12] S. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, hep-th/9609239

[13] K. Hori, H. Ooguri and C. Vafa, *Non-Abelian conifold transitions and N=4 dualities in three-dimensions*, hep-th/9705220

[14] K. Intriligator and N. Seiberg, Phys. Lett. $\underline{B387}$ (1996) 513

[15] S. Katz and C. Vafa, *Matter from geometry*, hep-th/9606086

[16] M. Bershadsky, V. Sadov and C. Vafa, Nucl. Phys. B463 (1996) 398;
A. Klemm and P. Mayr, Nucl. Phys. $\underline{B469}$ (1996) 37 ;
S. Katz, D. R. Morrison and M. R. Plesser, Nucl. Phys. $\underline{B477}$ (1996) 105;
P. Berglund, S. Katz, A. Klemm, P. Mayr, Nucl. Phys. $\underline{B483}$ (1997) 209.

[17] M. Bershadsky et. al., Nucl. Phys. $\underline{B481}$ (1996) 215

[18] R. Miranda, *Smooth Models for Elliptic Threefolds*, in R. Friedman and D. R. Morrison, editors, "The Birational Geometry of Degenerations", Birkhäuser, 1983

[19] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F theory*, hep-th/9701162

[20] P. Aspinwall and M. Gross, Phys. Lett. $\underline{B387}$ (1996) 735

[21] M. Bershadsky et. al., Nucl. Phys. $\underline{B481}$ (1996) 215

[22] S. Katz, P. Mayr and C. Vafa, to appear.

[23] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press 1990

[24] N. Berkovits and W. Siegel, Nucl. Phys. $\underline{BB462}$ (1996) 213.

[25] A. Strominger, Nucl. Phys. $\underline{B451}$ (1995) 96

[26] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. $\underline{B361}$ (1995) 59

[27] P. Aspinwall and D. R. Morrison, *String theory on K3 surface*,hep-th/9404151

[28] K. Kodaira, Annals of Math., Vol. $\underline{77}$, No. 3 (1963)

[29] E. Witten, *Solutions of four-dimensional field theories via M theory*, hep-th/9703166

[30] K. Landsteiner, E. Lopez and David A. Lowe *N=2 supersymmetric gauge theories, branes and orientifolds*, hep-th/9705199;
A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, *M theory and Seiberg-Witten curves: Orthogonal and symplectic groups*, hep-th/9705232

[31] K. Dasgupta and S. Mukhi, Phys. Lett. $\underline{B385}$ (1996) 125

[32] A. Sen, Nucl. Phys. $\underline{B475}$ (1996) 562

[33] C. Vafa, *Topological mirrors and quantum rings*, in *Essays on mirror manifolds*, edited by S.-T. Yau, International Press 1992.

[34] E. Witten, Nucl. Phys. $\underline{B403}$ (1993) 159

[35] P. Aspinwall, B. R. Greene and D. R. Morrison, Nucl. Phys. $\underline{B416}$ (1994) 414

[36] D. R. Morrison and M. R. Plesser, Nucl. Phys. $\underline{B440}$ (1995) 279

[37] P. Candelas and A. Font, *Duality between the webs of heterotic and type II vacua*, hep-th/9603170

[38] P. Candelas, E. Perevalov and G. Rajesh, *Toric geometry and enhanced gauge symmetry of F theory / heterotic vacua*, hep-th/9704097;
E. Perevalov and H. Skarke, *Enhanced gauged symmetry in type II and F theory compactifications: Dynkin diagrams from polyhedra*, hep-th/9704129

[39] V. Batyrev, J. Alg. Geom. $\underline{3}$ (1994) 493

[40] D. R. Morrison and M. R. Plesser, *Towards mirror symmetry as duality for two-dimensional abelian gauge theories*, hep-th/9508107

[41] B. Greene, C. Vafa and N. Warner, Nucl. Phys. $\underline{B324}$ (1998) 371

[42] E. Martinec, *Criticality, Catastrophe and Compactifications*, V.G. Knizhnik memorial volume, 1989.

[43] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Comm. Math. Phys. <u>167</u> (1995) 301

[44] P. Aspinwall, B. R. Greene and D. R. Morrison, Nucl. Phys. <u>B420</u> (1994) 184

[45] J.M. Isidro, A. Mukherjee, J.P. Nunes and H.J. Schnitzer, *A new derivation of the Picard-Fuchs equations for N=2 Seiberg-Witten theories*, hep-th/9609116; *A note on the Picard-Fuchs equations for N=2 Seiberg Witten Theories*, hep-th/9703176; *On the Picard-Fuchs equations for massive N=2 Seiberg- Witten theories*, hep-th/9704174

[46] M. Alishahiha, *On Picard-Fuchs equations of the SW models*, hep-th/9609157; *Simple derivation of the Picard-Fuchs equations for the Seiberg-Witten models*, hep-th/9703186

[47] V. Batyrev and L. A. Borisov, *On Calabi–Yau complete intersections in toric varieties*, alg-geom/9412017

[48] V.V. Batyrev, Duke Math. J. **69** (1993) 349

[49] V. Batyrev and D. Cox, Duke Math. J. **75** (1994) 293

[50] See e.g., V. Arnold, A. Gusein-Zade and A. Varchenko, *Singularities of Differentiable Maps I, II*, Birkhäuser 1985.

[51] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F theory*, hep-th/9701165

[52] E. Looijenga, Invent. Math. **38** (1977) 17; Invent. Math. **61** (1980) 1.

[53] P. Argyres, *S-Duality and Global Symmetries in $N = 2$ Supersymmetric Field Theory*, hep-th/9706095

[54] M. R. Douglas, S. Katz and C. Vafa, *Small instantons, Del Pezzo surfaces and type I' theory*, hep-th/9609071

[55] D. R. Morrison and N. Seiberg, Nucl. Phys. <u>B483</u> (1997) 229

[56] K. Intriligator, D. R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, hep-th/9702198

[57] M. Bershadsky and C. Vafa, *Global anomalies and geometric engineering of critical theories in six-dimensions*, hep-th/9703167

[58] J. D. Blum and K. Intriligator, *Consistency conditions for branes at orbifold singularities*, hep-th/9705030; *New phases of string theory and 6-D RG fixed points via branes at orbifold singularities*, hep-th/9705044

[59] P. Aspinwall and D. R. Morrison, *Point - like instantons on K3 orbifolds*, hep-th/9705104

[60] A. Lawrence and N. Nekrasov, *Instanton sums and five-dimensional gauge theories*, hep-th/9706025

[61] J. de Boer, K. Hori, H. Ooguri and Y. Oz, *Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*, hep-th/9611063; *Mirror symmetry in three-dimensional theories, SL(2,Z) and D-brane moduli spaces*, hep-th/9612131

[62] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*, hep-th/9611230

[63] J. A. Minahan and D. Nemeschansky, Nucl. Phys. $\underline{B482}$ (1996) 142; Nucl. Phys. $\underline{B489}$ (1997) 24;
O. Ganor, Nucl. Phys. $\underline{B488}$ (1997) 223;
O. Ganor, D. R. Morrison and N. Seiberg, Nucl. Phys. $\underline{B487}$ (1996) 93;
W. Lerche, P. Mayr and N. P. Warner, *Noncritical strings, Del Pezzo singularities and Seiberg- Witten curves, hep-th/9612085*

[64] E. Witten, Nucl. Phys. $\underline{B471}$ (1996) 195

[65] D. R. Morrison and C. Vafa, Nucl. Phys. $\underline{B476}$ (1996) 437

[66] D. R. Morrison and C. Vafa, Nucl. Phys. $\underline{B473}$ (1996) 74